PLANE

TRIGONOMETRY

A TEXTBOOK FOR SECONDARY SCHOOLS

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PLANE TRIGONOMETRY

Dr. M. K. Singal
Professor & Head,
Department of Mathematics
Meerut University

and

Asha Rani Singal
Lecturer in Mathematics
Meerut University

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The third chapter deals with *addition theorems for trigonometric functions*. In this chapter, the addition theorem for the cosine has been established for all angles by using the unit circle, and all the other addition theorems have been deducted from it. The formulae for trigonometric functions of associated angles have been deduced from the addition formulae, thus saving a lot of labour that is usually spent in giving geometrical proofs for each set of formulae. The chapter closes with a section on the use of tables of trigonometric ratios.

The fourth chapter deals with *properties of triangles*. Though much of the material presented in this chapter is classical, yet an attempt has been made to cut down geometrical proofs to a minimum. The cosine formula has been proved by the analytic method and many of the remaining formulae have been deduced from it.

The fifth chapter deals with *logarithms*. After introducing the logarithmic function as the inverse of the exponential function, methods of solving logarithmic and exponential equations have been discussed. While introducing the use of logarithmic tables, the role of interpolation has been emphasized. While dealing with the applications of logarithms, problems have been drawn from various fields.

The sixth chapter deals with *solution of triangles and its applications*. While dealing with solution of triangles, the importance of arranging the working in a neat and systematic form and the degree of accuracy that can be obtained with a given data have been stressed. In problems dealing with heights and distances, an attempt has been made to give realistic problems. Problems involving oblique triangles have been given a due place. Problems involving angles other than 30°, 45°, and their integral multiples are also given.

The seventh chapter deals with *trigonometric functions of a real variable*. The first half of the chapter contains material which is not at all taught at present but is so important that it must find a place in the curriculum. After defining trigonometric functions of numbers, some of their basic properties such as periodicity, intervals of monotony, range, boundedness (or otherwise) are discussed. The second half of the chapter is devoted to graphing. Besides the graphs of the trigonometric functions, graphs of equations of the form $y = a \sin (bx + c)$ have also been discussed. Graphing by composition of ordinates has also been briefly discussed.

The eighth chapter deals with *inverse trigonometric functions*. Inverse functions constitute a rather important but badly neglected topic in mathematics. In this chapter an attempt has been made to bring home to the students as to what do we mean by inverse functions and as to under what conditions can a function possess an inverse. After dealing briefly with some basic facts about inverse functions in general, the inverse trigonometric functions have been defined and studied.

The ninth chapter deals with *trigonometric equations*. This chapter is, in a certain sense, a continuation of the fifth chapter. In this chapter, methods of solving trigonometric equations, inverse trigonometric equations, and trigonometric inequations have been discussed. It has been clearly pointed out that while the discussion relates to trigonometric functions of numbers, the same could be applied to trigonometric functions of angles as well.

The problems are properly graded and have been designed to test both the understanding of the subject and the computational skill of the student. A large number of problems dealing with the theory are also given. At the end of each chapter, a set of assorted exercises based on the material discussed in the chapter has also been given.

The authors are grateful to the members of the editorial board for their valuable suggestions from time to time. They are specially grateful to Prof. J. N. Kapur for his kind guidance throughout the preparation of the book. Thanks are also due to Miss Asha Singal, Miss Asha Mathur, Miss Pushpa Jain and Shri P. N. Gupta for checking the exercises.

It is hoped that the book will prove useful to those for whom it is intended. Suggestions for the improvement of the book will be gratefully accepted.

M. K. SINGAL
ASHA RANI SINGAL

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Chapter 1

Basic Concepts

1.1. Sets

In our daily life we come across collections of various types. A deck of cards, a bunch of grapes, a herd of cattle, a class of students, a team of players, an army of soldiers, are all examples of such collections. The words deck, bunch, herd, class, team and army all denote collections. In mathematics also we have to deal with collections. Since mathematicians are wise and simple people, therefore (in order to simplify matters), they have given a very simple name to all *well-defined* collections that they have to deal with. They call such a collection a *set*. Thus:

A set is a well-defined collection of objects.

Each object belonging to a set is called an *element* (or a *member*) of the set. We generally use capital letters A, B, C, S, etc., to denote sets and small letters such as a, b, c, s, etc., to denote elements of a set. The symbol $\in$ is used to indicate 'belongs to' (or 'is an element of'). Thus 'x is an element of S' is written as $x \in S$. The symbol $\notin$ is used to indicate 'does not belong to' (or 'is not an element of'). Thus $x \notin S$ means that 'x does not belong to S'.

Let us try to understand the phrase 'well-defined collection' carefully. When we say that a collection is well-defined we mean that we can say without any ambiguity whatsoever as to what the members of the collection are. Given any object, we should be able to say whether it belongs to the collection or not. If a collection does not pass this test successfully, it cannot be called a set.

There are several ways of describing a set. We may describe a set by listing all its elements. For example, the set V of vowels in the English alphabet is

$$V = \{ a, e, i, o, u \} ;$$

the set P of all prime ministers of India is

$$P = \{ \text{Nehru, Shastri, Indira} \} ;$$

the set D of all divisors of 12 is

$$D = \{ 1, 2, 3, 4, 6, 12 \} .$$

Sometimes a set has too many elements so that listing all the elements is tedious, but after recognizing a few elements we can understand what all the elements are. Thus for example,

$$1, 2, 3, \ldots\ldots\ldots\ldots\ldots\ldots, 100$$

is the set of all natural numbers up to (and including) 100. The dots indicate that 4, 5, 6, etc. are all members of the set.

Sometimes it is not possible to list all the elements of the set ; but from a few elements we can follow without any ambiguity all the elements of the set. For example $E = \{ 2, 4, 6, 8, \ldots\ldots \}$ is the set of all positive even integers.

A set can be described by some property that is characteristic of all the elements of the set. By saying that the property must be 'characteristic' we mean that it should be possessed by all the elements of the set and that any object possessing the property must belong to the set. This method of describing sets is called the 'Set-builder Method'. Following are some examples of sets described by this method :

(i) {n : n is a *natural* number }.

(ii) {p : n is a *prime* number }.

The symbol ' : ' stands for 'such that'.

Equality of sets. Two sets are said to be equal if and only if they have the same elements. Also the order in which the elements of a set are written in any description is immaterial and repetitions of any element in a description do not alter the set in any manner. Thus :

(i) $\{3,4,7\} = \{4,3,7\}$.

(ii) $\{1,2,5\} = \{1,3,5\}$.

(iii) $\{2,2,3,4\} = \{2,3,4\}$.

The empty set. Sometimes we have to talk of a set that does not contain any element. For example, the set of negative integers greater than 2 has no element. A set having no element is called the *empty set* or *null set*. The symbol ϕ is used to denote the empty set. A set is said to be non-empty if it is not equal to ϕ . Let us note here that the sets $\{0\}$, $\{\phi\}$ are both different from the empty set. In fact, $\{0\}$ is a set whose only element is 0; $\{\phi\}$ is a set whose only element is the empty set.

Subsets. We often consider sets whose elements are also the elements of another set. For example, let M be the set of all ministers of the union government and let P be the set of all members of the parliament. Then every member of M is also a member of P. We express this by saying that M is a subset of P. Again, let V be the set of all vowels in the English alphabet and let A be the set of all letters in the English alphabet. Then every member of V is also a member of A. We express this by saying that V is a subset of A. We say that a set A is a subset of a set B if every element of A is also an element of B. In symbols, this fact is indicated by writing $A \subset B$.

If A is a subset of B then we also say that B is a superset of A. We express this fact by writing $B \supset A$.

The empty set is, by convention, a subset of every set, i.e., if A be any set, then $\phi \subset A$.

If A, B be any two sets, then we write $A \not\subset B$ to denote that A is not a subset of B.

Example :

Let $A = \{1,2,3\}$, $B = \{1,2,3,4\}$, $C = \{2,3,4\}$. Then $A \subset B$, $A \not\subset C$, $B \supset A$, $A \not\supset C$, $C \not\subset A$.

The power set. The set of all subsets of a given set X is called the *power set* of X and is denoted by ρ (X). For example, if $X = \{1,2,3,\}$, then $p(X) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, X\}$. Let us observe that in this case X has 3 elements and p (X) has $8 = 2^3$ elements. It can be proved that if a set contains n elements, then its power set contains 2^n elements (prove it).

1.2. Real Numbers

We shall now deal with one of the most important sets, the set R of real numbers. The numbers 1,2,......,—1,—2,........., 1/2,2/3,........., $\sqrt{2}$, $\sqrt[3]{4}$, π etc. are all members of **R.** The set **R** has many pleasant properties that make it both useful and interesting. We shall state some of these properties.

A 1. Closure. The set **R** is closed under addition. This means that if a and b are two real numbers, then there is a unique real number c such that $a+b = c$.

A 2. Associative law. The operation of addition is *associative.* That is, if a,b,c, are any three real numbers, then $(a+b)+c = a+ (b+c)$.

A 3. Commutative law. The operation of addition is *commutative.* That is, if a and b are any two real numbers, then $a+b = b+a$.

A 4. Neutral element. The real number 0 is a neutral element for addition. That is, if a be any real number, then $a+0=0+a = a$.

A 5. Negative. For each real number a there is a unique real number b (called the negative of a) such that $a+b = b+a=0$.

M 1. Closure. The set of real numbers is closed under multiplication. That is, if a and b are any two real numbers, then there is a unique real number c such that $ab = c$.

M 2. Associative law. The operation of multiplication is *associative*. That is, if a, b and c are any three real numbers, then $a(bc) = (ab)c$.

M 3. Commutative law. The operation of multiplication is *commutative*. That is, if a and b are any two real numbers, then $ab = ba$.

M 4. Identity. The real number 1 is an identity for multiplication. That is, if a be any real number, then $a.1 = 1.a = a$.

M 5. Inverse. For each real number a other than 0, there is a unique real number b, (called the inverse of a) such that $ab = ba = 1$.

D. Distributive law. Multiplication distributes itself over addition. That is, if a, b and c are any three real numbers, then $a(b+c)=ab+ac$.

O 1. For each pair of real numbers a and b, exactly one of the following holds :

$$a < b,\ a = b,\ \text{or } b < a.$$

O 2. If a, b, c be real numbers such that $a < b$ and $b < c$, then $a < c$.

O 3. If a, b, c be real numbers such that $a < b$, then $a+c < b+c$.

O 4. If a, b, c be real numbers such that $a < b$, $0 < c$, then $ac < bc$.

C. Completeness. To state this property we shall start by defining a new term. A real number k is said to be an upper bound of a subset A of **R**, if k is not exceeded by any member of A. Also, we then say that A is bounded above. It is not necessary that every subset of **R** should have an upper bound. The point of interest is, however, the fact that :

Every set of real numbers that is bounded above has a least upper bound. That is, if $A \subset \mathbf{R}$ be bounded above, then the set of upper bounds of A has a smallest member.

The properties that we have listed above are briefly described by saying that *the set of real numbers is a complete ordered field.*

The above properties characterize the set of real numbers. That is, the set of real numbers is the *only* set that shares these properties and that is why we have chosen these properties, in preference to scores of other properties that it possesses.

The following are some important subsets of **R** :

(i) The set $\mathbf{N} = \{1,2,3,\ldots\ldots\ldots\ldots\}$ of *natural numbers.*

(ii) The set $\mathbf{Z} = \{\ldots\ldots\ldots,-2,-1,0,1,2,\ldots\ldots\ldots\}$ of *integers.*

(iii) The set $\mathbf{Q} = \{p/q : p,q \in \mathbf{Z} \text{ and } q \neq 0\}$ of *rational numbers.*

We already know that

$$N \subset Z \subset Q \subset R.$$

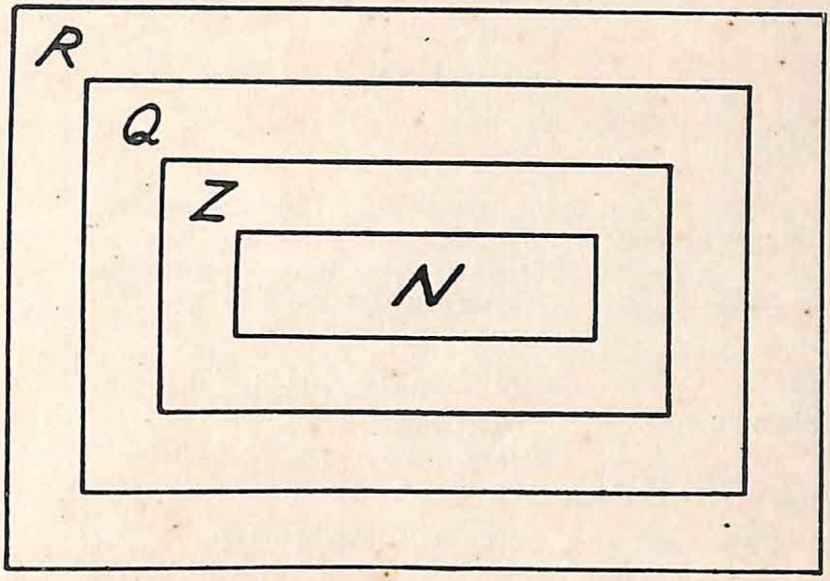


Fig. 1.1

1.3. Statements

A sentence, of which it is meaningful to say, whether it is true or false, is called a *statement*. For example, each of the following sentences is a statement :

- (i) The earth revolves about the sun.
- (ii) New Delhi is the capital of India.
- (iii) Ramanujan was a great mathematician of India.
- (iv) 7 is less than 10.
- (v) 6 is a prime number.

The statements (i) – (iv) are true statements but (v) is a false statement. In mathematics we make statements about objects that we study. In good mathematics we try to avoid false statements.

Compound statements. Consider the following statements :

- (a) Asha is intelligent and industrious.
- (b) He is a fool or a liar.
- (c) If two triangles are congruent, then they are equal in area.
- (d) $m+4$ is an even integer if and only if m is an even integer.

Each of the above statements consists of two statements. Such statements are known as *compound statements*.

Let us consider the above statements one by one. The statement (a) says that Asha is intelligent and Asha is industrious. Here two statements have been connected by the connective 'and'. In mathematics a compound statement obtained by connecting two statements by the connective 'and' is called a *conjunction*. The symbol '$\wedge$' is used for the connective 'and'. Thus we can write the above statement as :

Asha is intelligent $\wedge$ Asha is industrious.

Thus 14 is a multiple of 7 $\wedge$ 14 is a divisor of 56, means that 14 is a multiple of 7 and a divisor of 56.

The statement (b) says about a certain person that he is a fool *or* he is a liar. Here two statements have been connected by the connective 'or'. A compound statement consisting of two statements connected by the connective 'or' is called a *disjunction*. We use the symbol '$\vee$' for disjunction. Thus we can write the above statement as :

He is a fool $\vee$ He is a liar.

Similarly, the compound statement

He is smart $\vee$ He is lucky,

means 'He is smart or lucky.'

The compound statement (c) consists of the statements 'two triangles are congruent' and 'two triangles are equal in area' connected by 'if ...... then'. Such a statement is called an *implication*. We use the symbol $\Rightarrow$ (read as 'implies' for 'if ....then'). Thus we may write the statement (c) as follows :

Two triangles are congruent $\Rightarrow$ they are equal in area.

Similarly, the statement

p^2 is even $\Rightarrow$ p is even

means that if p^2 is even, then p is even and is read as 'p^2 is even *implies* p is even.' The compound statement (d) consists of the statements '$m+4$ is an even integer' and 'm is an even integer' connected by the phrase 'if and only if'. Such a statement is called a *double implication*. We use the symbol '$\Longleftrightarrow$' (read as 'implies and is implied by' (or 'is equivalent to') for 'if and only if'. Thus we may write the statement (d) as follows :

$m+4$ is an even integer $\Longleftrightarrow$ m is an even integer.

Similarly, the statement

p^2 is even $\Longleftrightarrow$ p is even

means that 'p^2 is even if and only if p is even'. Two statements connected by '$\Longleftrightarrow$' are said to be *equivalent*.

Open statements and truth sets. We often make statements about members of a set. For example, if S is the set of students in a certain higher secondary school, then the statement 'he is in class X' is true for some members of S and false for other members of S. Similarly, if T is the set of all female members of the parliament then the statement 'she is a minister' is true for some members of T and false for other members. Again, if **Z** is the set of all integers, '$x \in \mathbf{Z} \wedge x^2=4$' is true when x is replaced by 2 or -2 and is false when x is replaced by any other member of **Z**.

In the above examples 'he', 'she' and 'x' all play similar roles. Each of these is a *variable*. A variable may represent any member of a given set. In the above examples we may replace 'he' by any member of S, 'she' by any member of T and 'x' by any member of **Z**.

A statement containing a variable is called an *open statement* and the set whose elements may serve as replacements for the variable is called the *domain* or the *replacement set* of the variable. The subset of the domain for which the open statement is true is called the *truth set* or *solution set* of the statement. Each member of the solution set is called a solution of the open statement. Thus for the last one of the above examples, **Z** is the domain of the variable, $\{-2, 2\}$ is the truth set of the open statement and each of the numbers -2 and 2 is a solution of the open statement.

1.4. Relations and Functions

We often use such partial statements as :

is the daughter of,
is the brother of,
is a friend of,
is greater than,
is a divisor of,
is similar to.

Each such partial statement describes a relation between two objects, one that precedes it and the other that follows it. For example, 'Satish is the son of Shri Ram Prakash' describes the relation of Satish to Shri Ram Prakash. A complete statement such as this is either true or false. Thus, for example, '7 is greater than 5' is true but '5 is greater than 7' is false. We can formulate the concept of relation about which we have talked above in an intutive manner, in the language of sets. Let us think of the relation 'is the daughter of' in set language.

Let D be the set of all female human beings, living or dead and let F be the set of all male human beings, living or dead. Then the relation 'is the daughter of' applies to some ordered pairs of the type (a, b) where $a \in D$, $b \in F$, and 'a is the daughter of b'. The set of all ordered pairs (a, b) for which the statement 'a is the daughter of b' is true is a subset of $D \times F$. If we are given this subset, then we know exactly as to for what pairs the statement '......... is the daughter of.........' is true. This suggests that we can talk of subsets of $D \times F$ as relations.

Definition. *A relation R on a set A to a set B is a subset of $A \times B$. If $a \in A$, $b \in B$ be such that $(a, b) \in R \subseteq A \times B$, then we say that a is R-related to b.*

A relation on a set A to itself is called a relation in the set A. Every subset of $A \times A$ is a relation in A.

If R be a relation, then the set of all first co-ordinates of members of R is called the domain of R and the set of all second co-ordinates of members of R is called the range of R. Thus

$$\text{dom. } R = \{ x : (x, y) \in R \},$$
$$\text{ran. } R = \{ y : (x, y) \in R \}.$$

Example : Let $A = \{1, 2, 3, 8\}$, $B = \{-1, 6, 9\}$. The set $A \times B$ consists of 12 elements. Since $p(A \times B)$ will have 2^{12} members, therefore we can have 2^{12} relations on A to B.

Let $R = \{(1, -1), (2, -1), (3, -1), (8, -1), (8, 6)\}$. Then R is the relation 'is greater than' on A to B.

Let $S = \{(2, 6), (3, 9)\}$. Then S is the relation 'is one-third' of on A to B.

Let $T = \{(1, 6), (1, 9), (2, 6), (2, 9), (3, 6), (3, 9), (8, 9)\}$. Then T is the relation 'is less than' on A to B.

Let $V = \{(1, 6), (2, -1)\}$. Then V is also a relation on A to B but we do not have a familiar expression for it in the form 'is the...........of'.

Functions. Consider the graphs of the relations $R = \{(-3, 2), (2, 1), (2, 3), (3, 1)\}$, and $S = \{(-3, 2), (1, 0), (2, 3), (3, 1)\}$. In the graph of R we find that the points corresponding to (2, 1) and (3, 1) lie on the same vertical line. This, however, is not true for the graph of S. No two points of the graph lie on the same vertical line. Such relations occupy an important place in mathematics. We call them functions.

A *function* is a relation which assigns to each element of the domain a unique element of the range.

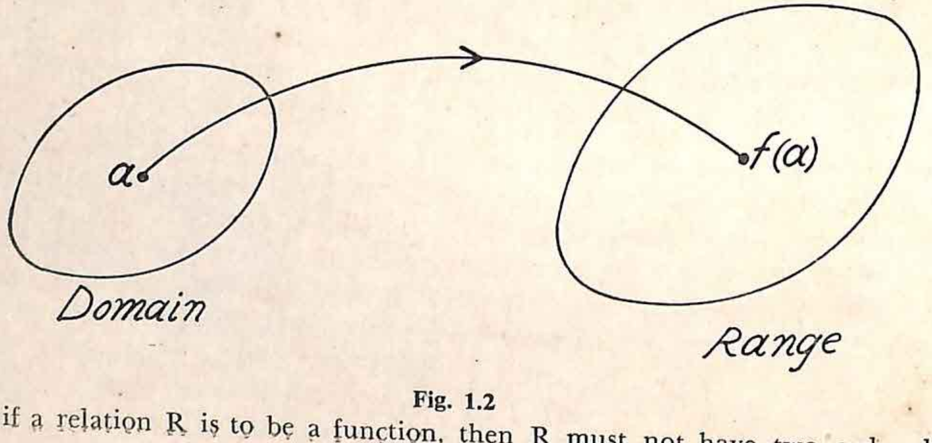


Fig. 1.2

Thus, if a relation R is to be a function, then R must not have two ordered pairs with the same first co-ordinate. (Of course it can have any number of ordered pairs with the same second co-ordinate.).

Since a function is nothing else but a set, therefore we may use the same notation for functions as for sets. However, it is more common to use the letters f and g to denote functions. For example, we may write

$$f = \{ (x, y) : y = 3x, x \in \mathbf{R} \}$$

to denote a function which pairs with each real number x its triple. To indicate the unique number in the range which this function associates with any number, say 4, in the domain, we use the symbol f(4) (read as f of 4 or the value of f at 4). Thus $f(4) = 3.4 = 12$. Similarly, $f(1)=3.1=3$, $f(-1/2)=3.(-\frac{1}{2})=-\frac{3}{8}$.

For any pair (x, y) in f, the second co-ordinate y is the value of f at x, and therefore we can also specify the function as

$$f = \{ (x, f(x)) : f(x)=3x, x \in \mathbf{R} \}.$$

We also call f(x) the image of x under f.

Since a function is a relation, the domain of a function f is the set $\{x : (x, y) \in f\}$ and the range is the set $\{ y : (x, y) \in f \}$. Throughout the present book we shall assume that the domain of any functions f whose values are indicated by a formula for f(x), is the set R of real numbers, unless stated otherwise.

Examples : **1.** $f = \{(x, f(x)) : f(x) = x \}$.
Here f assigns to each element x of the domain, the same element, namely x. This function is called the identity function on R.

2. $f = \{ (x, f(x)) : f(x) = 4 \}$. Here the range is the one elementic set $\{ 4 \}$ Such a function is called a constant function.

3. $f = \{ (x, y) : y = x+1 \}$. Here f assigns to distinct members in the domain, distinct members in the range.

4. $f = \{ (x, y) : y = x^2 \}$. Here f assigns to the numbers 3 and -3 in the domain the same number, viz. 9, in the range.

5. $f = \{ (x,f(x)) : f(x) = 1/x, x \in R^* \}$. where R^* is the set of all non-zero real numbers.

One-to-one functions. Consider the following functions :

(i) $f = \{ (1,2), (3,2), (-2,4) \}$.
(ii) $g = \{ (1,2), (3,-1), (-2,4) \}$.

For the function f, $f(1) = f(3) = 2$, that is, two elements of the domain have the same image. For the function g, no two members of the domain have the same image.

A function in which no two members of the domain have the same image is called a *one-to-one function*. Thus g is a one-to-one function while f is not such a function. The functions in examples 1,3,5 above are one-to-one functions. The functions in examples 2 and 4 above are not one-to-one.

1.5. Angles

A straight line drawn from a fixed point and extending infinitely far in one direction is called a *ray*. The fixed point is called the *vertex* of the ray. Any point on a line, such as the point V in figure, separates the line into two rays each of which has V as the vertex. A ray can be specified by indicating first the vertex of the ray and then any other point of the ray. The two rays in the figure may thus be called VP and VQ respectively.

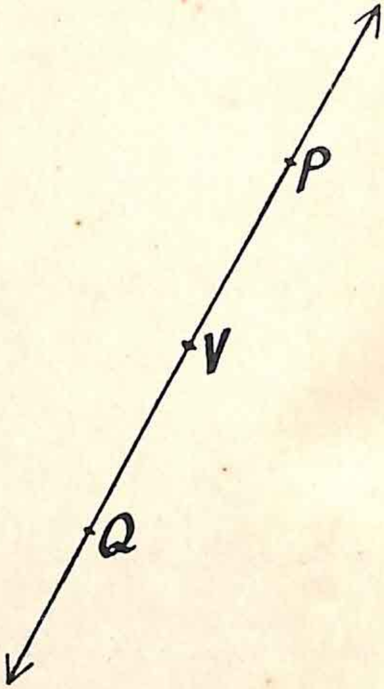


Fig. 1.3

The set of points constituting two rays with a common vertex together with a rotation which takes one of the rays into the other is called an *angle* (or a *directed angle*).

Consider an angle consisting of two rays OA and OB, having a common vertex O, and a rotation that takes OA into OB (figure 1.4).

The ray OA is called the *initial side* (ray) of the angle, the ray OB is called the terminal side of the angle and the common vertex O of the two rays is called the *vertex* of the angle.

Positive and negative angles. An angle that corresponds to a counter-clockwise rotation is called a *positive angle* and an angle that corresponds to a clockwise rotation is called a *negative angle*. Thus, for example, the angles in figures 1.4 (a) and (b) are positive angles and the angle in figure 1.4 (c) is a negative angle. An angle corresponding to a zero rotation is called a *zero angle*. The initial and terminal sides of a zero angle coincide. However, if the initial and terminal sides of an angle coincide, it does not necessarily mean that the angle is a zero angle, for, the initial and terminal sides coincide even in the case of an angle corresponding to any number of complete revolutions.

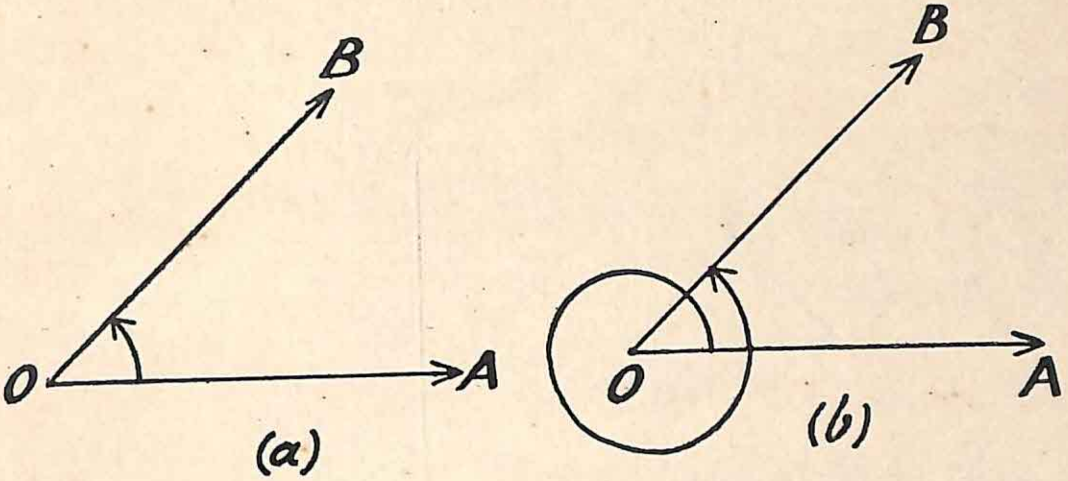


Fig. 1.4

Co-terminal angles. From the above discussion it is clear that there exist infinitely many angles having the same vertex, the same initial side and the same terminal side. All such angles differ from one another by a number of complete revolutions.

Angles having the same vertex, initial side and terminal side are said to be *co-terminal angles.*

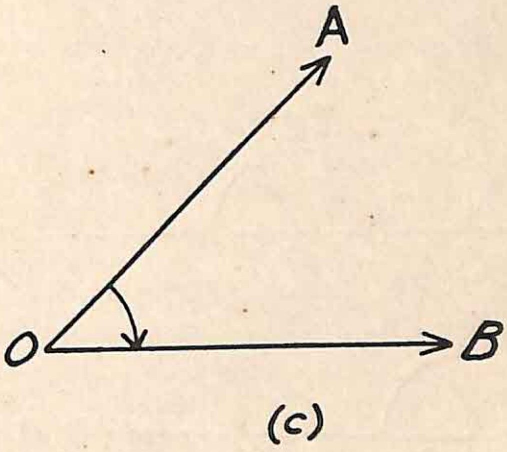


Fig. 1.4

1.6. Units of Measurements.

There are several units in which angles are measured. We shall describe two of them which are most commonly used.

Sexagesimal system. In this system, the primary unit of angle measure is a *degree,* written as 1°. If 1/360 of a revolution in the counter-clockwise direction takes the initial side into the terminal side then we say that the angle has a measure of one degree. A degree is divided into minutes and a minute is divided into seconds. One minute (written as 1′) is equal to 1/60 of a degree, and one second (written as 1″) is equal to 1/60 of a minute. An angle whose measure is 90° is called a *right angle.*

Angles whose measures are 45°, 90°, 180°, 270°, 360°, 420°, —30°, —180°, —420°, —360°, are shown in figure 1.5.

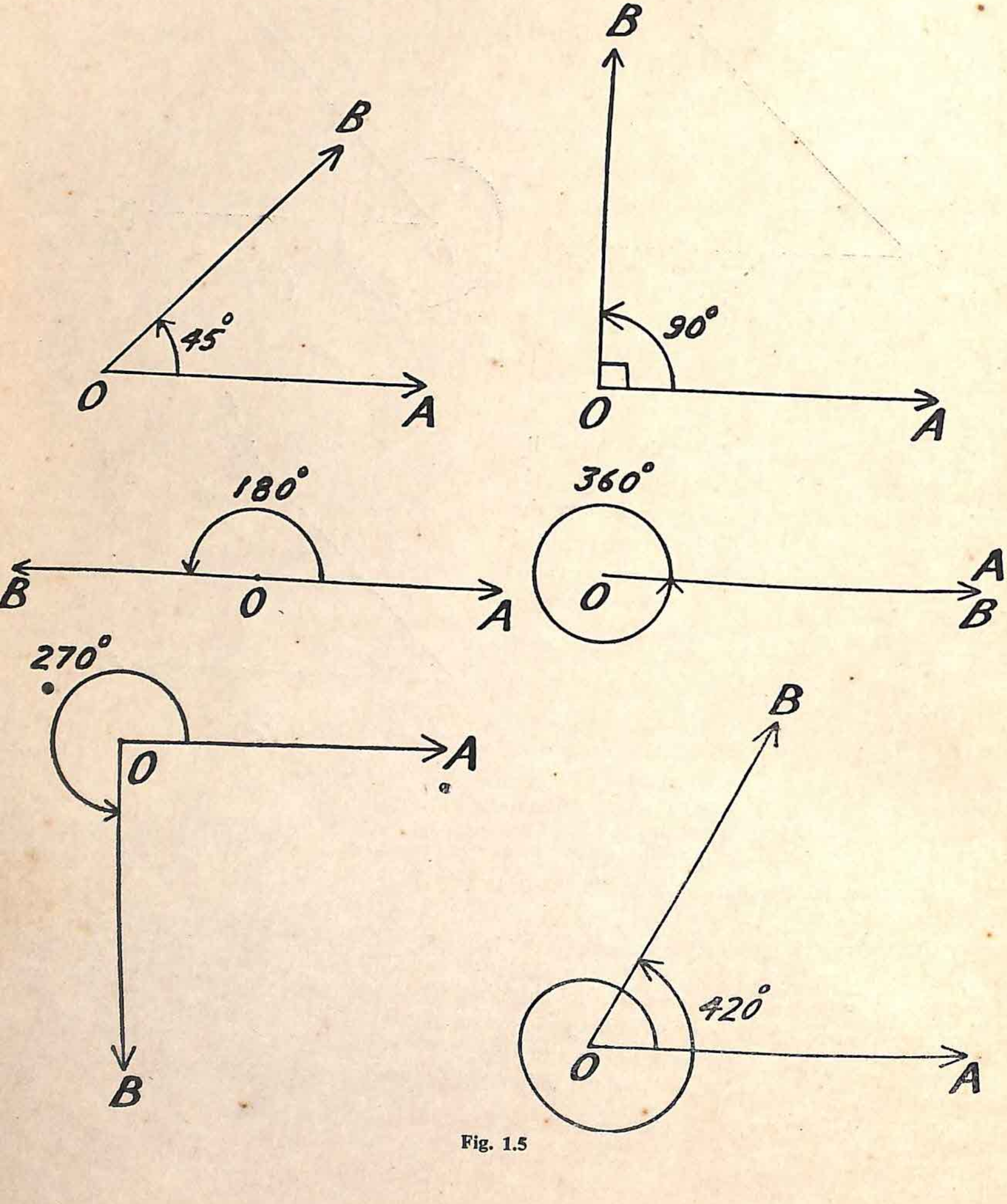
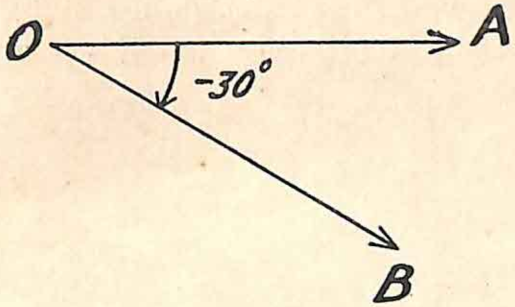
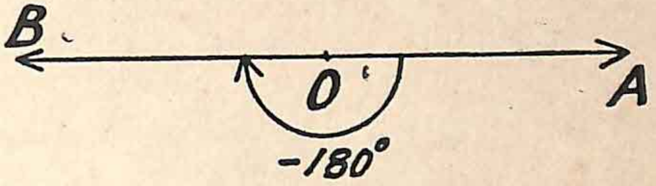
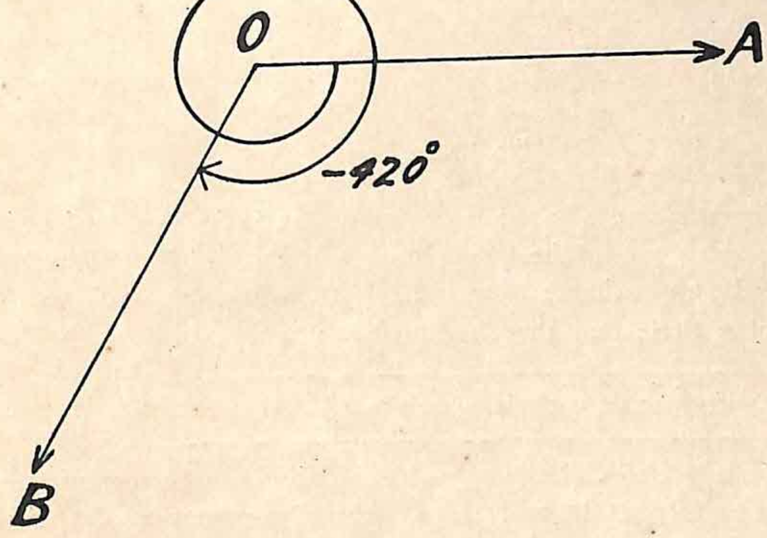


Fig. 1.5







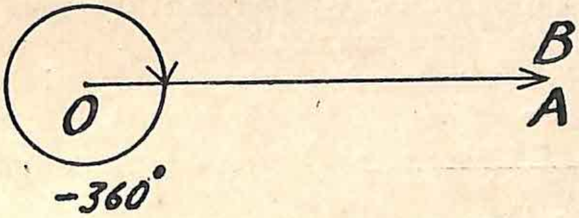


Fig. 1.5

The sexagesimal system is commonly used in engineering, astronomy, navigation, surveying, etc.

Example 1. Reduce $15^\circ\cdot31$ to degrees, minutes and seconds.

Solution. $0^\circ\cdot31 = \cdot31\times60' = 18'\cdot6.$

$0'\cdot6 = 6\times60'' = 36''.$

Therefore

$15^\circ\cdot31 = 15^\circ18'36''.$

Example 2. Reduce $31^\circ25'12''$ to degrees.

Solution.

$12'' = (12/60)' = 0'\cdot2.$

$25'\cdot2 = 25\cdot2/60$ degrees $= 0^\circ\cdot42.$

Therefore $31^\circ25'12'' = 31^\circ\cdot42.$

Circular System. In this system, the unit of measurement is a *radian*. A positive angle with its vertex at the centre of a circle which intercepts an arc equal in length to the radius of the circle is said to have a measure of one radian (written as 1^c). The following figure shows angles whose measures are 1 radian and $1\frac{1}{2}$ radians respectively.

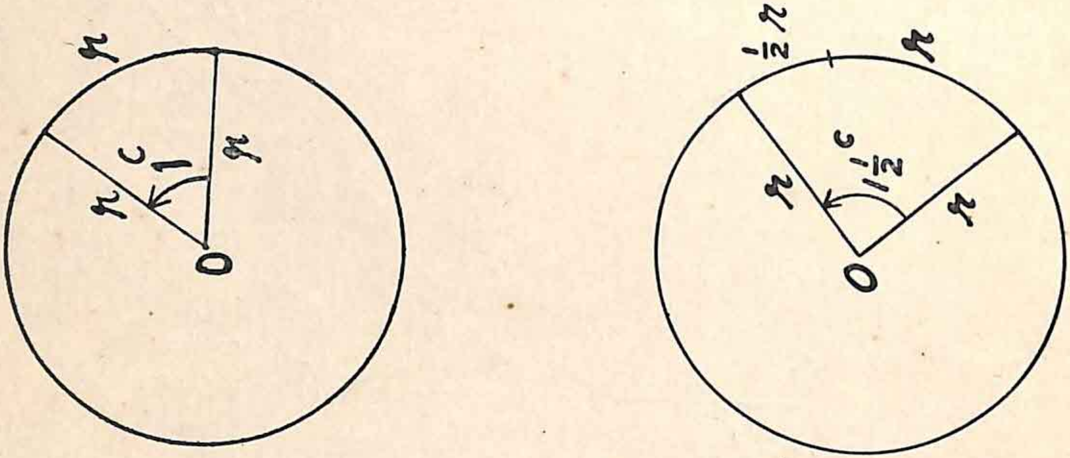


Fig. 1.6

The ratio of the circumference of a circle to its radius is known to be the same for all circles. It is denoted by the Greek letter π (read as 'pi'). This ratio is an irrational number. Its approximate value is 3·14159. The above result regarding the circumference of a circle may then be stated as :

$$\boxed{\text{Circumference of a circle of radius } r = 2\pi r.}$$

This means that for any circle, the length of one radius can be measured off 2π times along the circumference. In other words the circumference of a circle (whatever its radius may be) subtends an angle of 2π radians at the centre of the circle. Hence a radian is a fixed unit of angle measurement.

It is well known that equal arcs of a circle subtend equal angles at the centre. Since an arc of length r subtends an angle whose measure is 1 radian, therefore an arc of length l will subtend an angle whose measure is l/r radians. Thus, if in a circle of radius r, an arc of length l subtends an angle θ^c at the centre, we have

$$\boxed{\theta^c = l/r.}$$

Relation between degrees and radians. Since the circumference of a circle subtends at the centre of the circle an angle whose measure is 2π units in radians and is 360 units in degrees, therefore

$$2\pi \text{ radians} = 360^\circ,$$

or

$$\boxed{\pi \text{ radians} = 180^\circ.}$$

The above relation enables us to express a radian in terms of degrees, and a degree in terms of radians. Thus

$$1 \text{ radian} = 180^\circ/\pi,$$
$$= 57^\circ 17' 44''\cdot 81.$$

Also $1^\circ = \pi/180$ radian,

$$= \cdot 017453293 \text{ radian},$$

where we have taken $\pi = 3\cdot 1415927$.

If we take $\pi = 3\cdot 142$ (which suffices for many ordinary purposes),

$$1 \text{ radian} = 57^\circ 18',$$

and $1^\circ = \cdot 07145$ radian.

Example 3. Express $7^\circ 30'$ in radians.

Solution. $7^\circ 30' = 7^\circ.5 = 7\cdot 5 \times \frac{\pi}{180}$ radians $= \pi/24$ radian,

$= \cdot 1317$ radian.

Example 4. Express 2·5 radians in sexagesimal measure correct to nearest minute.

Solution. $2\cdot 5$ radians $= 2\cdot 5\,(57^\circ 17' 45'')$.

$= 143^\circ 14'$, to the nearest minute.

Radian measures of some common angles are given in the following table.

Degrees	30°	45°	60°	90°	180°	270°	360°
Radians	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π	$^3\pi/2$	2π

Example 5. Find the radius of a circle in which a central angle of 50° intercepts an arc of 150 cm.

Solution. Here $l = 150$ cm., $\theta = 50\pi/180$.

Hence by $r = l/\theta$, $r = 180 \times 3/\pi$ cm,

$= 1677$ cm (Taking $\pi = 22/7$).

Example 6. The minute hand of a watch is 1·5 cm long. How far does its tip move during a class period of 50 minutes ?

Solution.

In 50 minutes, the minute hand of a watch turns through 5/6 of a revolution, or $(5/3)\ \pi$ radians. Hence the required distance travelled is ( ).

$$\begin{aligned} s &= 1{\cdot}5\times 5\ \pi/3 \text{ cm} = 5\ \pi/2 \text{ cm}, \\ &= 5\times 3{\cdot}142/2 \text{ cm.} \qquad (\pi = 3{\cdot}142) \\ &= 7{\cdot}855 \text{ cm.} \end{aligned}$$

Example 7. Find the radius of a wheel whose circumference is 110 cm ($\pi = 22/7$).

Solution.

$$\begin{aligned} \text{Here} \quad 2\pi r &= 110 \text{ cm}, \\ \text{or} \quad (44/7)r &= 110 \text{ cm}, \\ \text{or} \quad r &= 7\times 110/44 \text{ cm} = 17{\cdot}5 \text{ cm}. \end{aligned}$$

1.7. Co-ordinate Plane. The Unit Circle

We introduce in the plane the rectangular co-ordinate axes X'OX, Y'OY. The direction OX is chosen as the positive direction of the x-axis and the positive direction of the y-axis is so chosen that a ray starting from OX coincides with the axis OY (y-axis) after revolving through a right angle in the positive (i.e. anti-clockwise) direction. Then the position of any point P is determined by two numbers x and y, called the co-ordinates of P where x = OM and y = ON, OM, ON being reckoned positive in the direction parallel to the positive x-axis and y-xis respectively (Figure 1·7). The plane with the co-ordinate axes introduced is referred to as the co-ordinate plane.

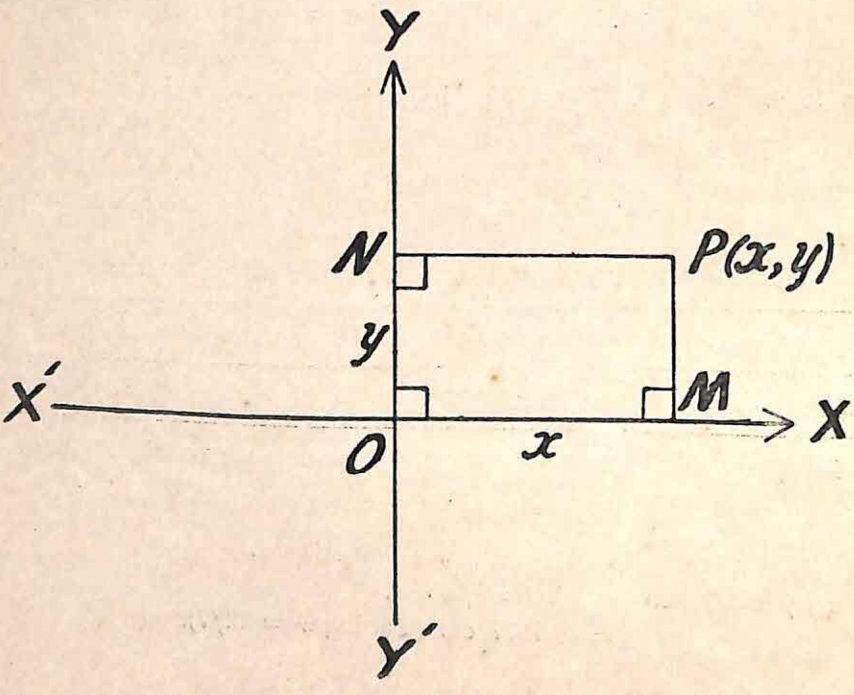


Fig. 1.7

A circle in the co-ordinate plane with its centre at the origin and unit radius is referred to as *the unit circle.* The co-ordinate axes divide the plane and along with it the unit circle in four quadrants (figure 1.8).

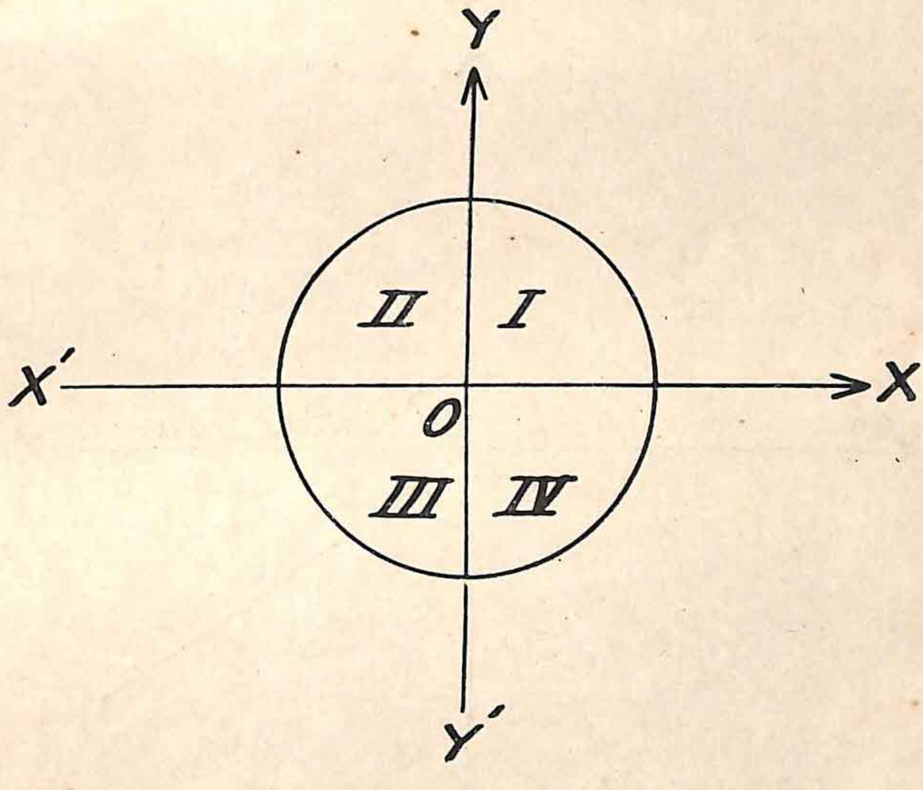


Fig. 1.8

Angles in standard position. In a co-ordinate plane, every point P other than the origin determines an infinite set of co-terminal angles each having the origin as the vertex, the positive half of the x-axis as initial side and the ray OP as the terminal side. Each such angle is said to be a position angle of P. If we know the measure of the position angle of a point P and the distance of P from the origin O, we can locate the position of P in the co-ordinate plane.

An angle whose vertex is the origin and whose initial side is the positive half of the x-axis is called an angle in the standard position. The angle XOP shown in figure 1.9 is an angle in standard position. Every position angle of any point P in the co-ordinate plane is an angle in standard position.

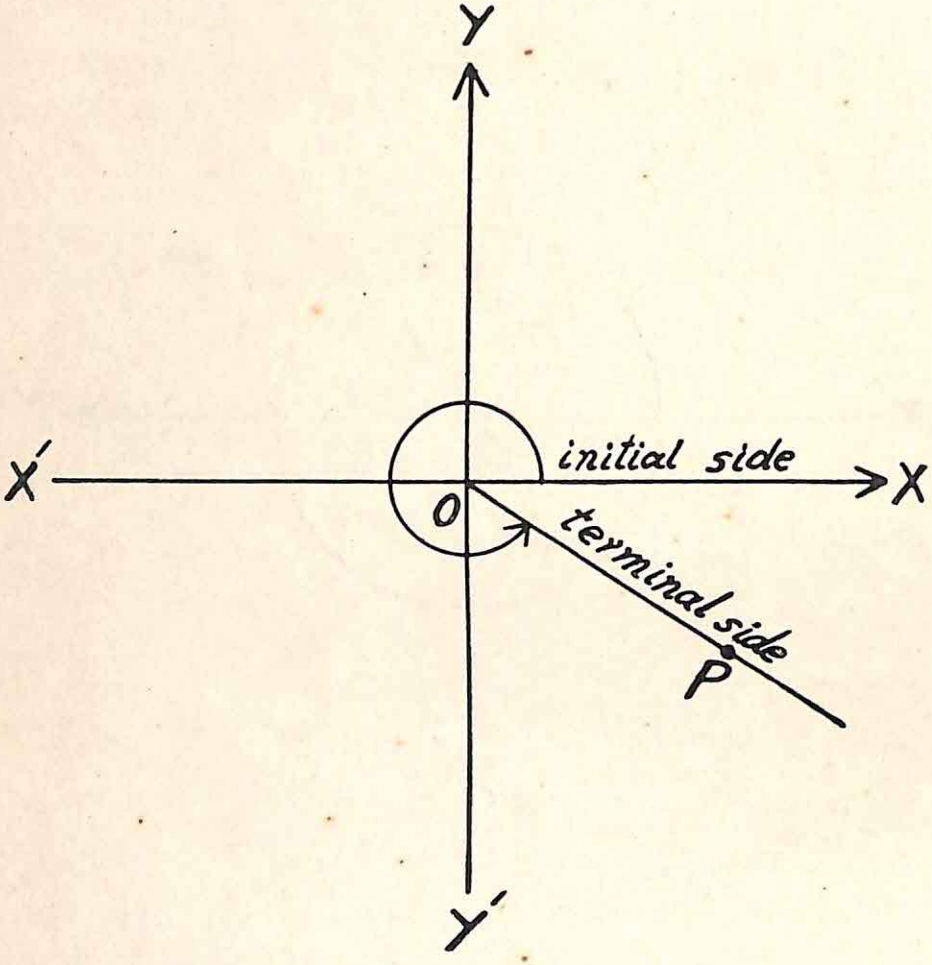


Fig. 1.9

Quadrantal angles. An angle in standard position whose terminal side coincides with any of the co-ordinate axes is called a quadrantal angle. Figure 1.10 shows some quadrantal angles.

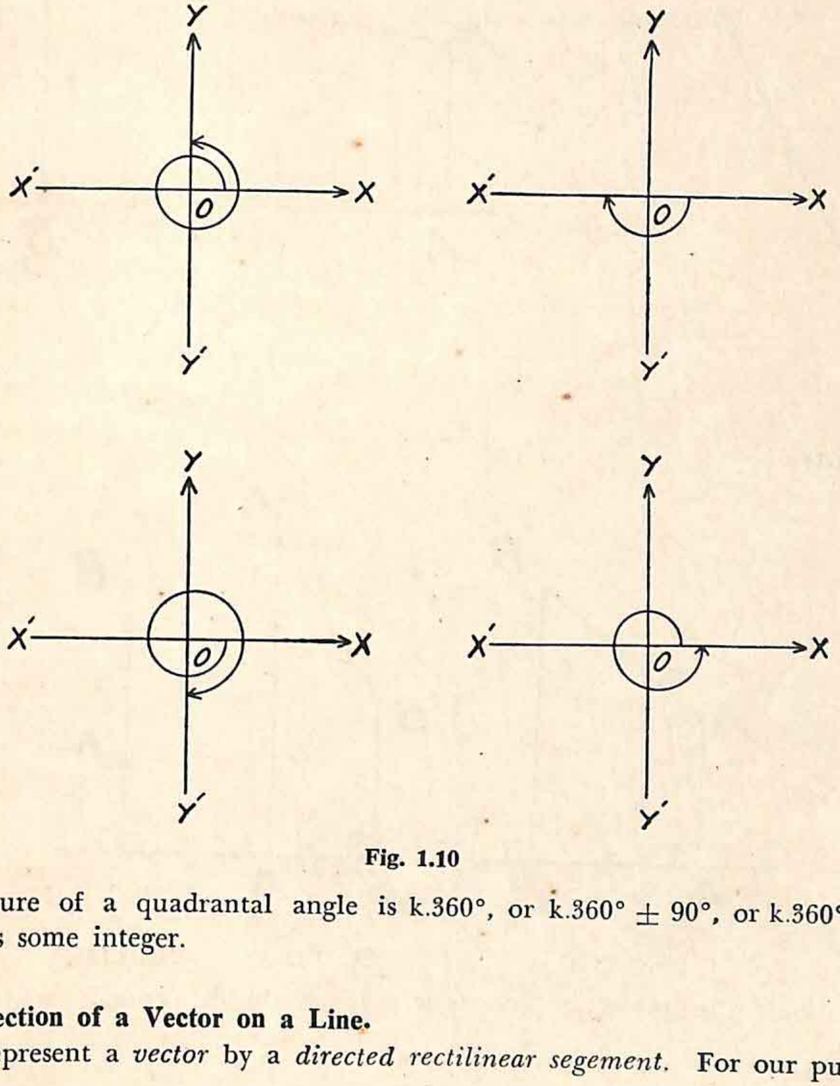


Fig. 1.10

The measure of a quadrantal angle is k.360°, or k.360° ± 90°, or k.360° ± 180°, where k is some integer.

1.8. Projection of a Vector on a Line.

We represent a *vector* by a *directed rectilinear segement.* For our purpose *we identify a vector with its representation.* By the symbol $\overrightarrow{AB}$ (figure 1.11) we mean a vector whose initial point is A and terminal point is B; $\overrightarrow{BA}$ is the vector with the initial point B and the terminal point A and we have by definition

$$\overrightarrow{AB} = -\overrightarrow{BA}$$

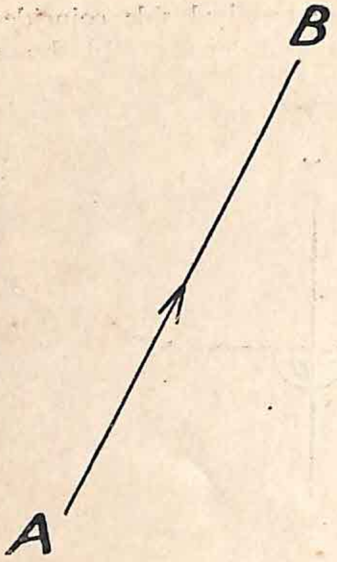


Fig. 1.11

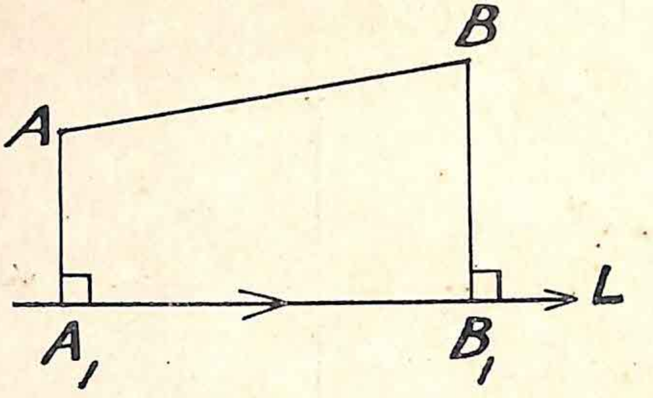


Fig. 1.12

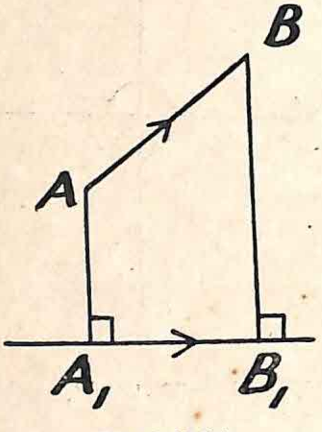


Fig. 1.13 (a)

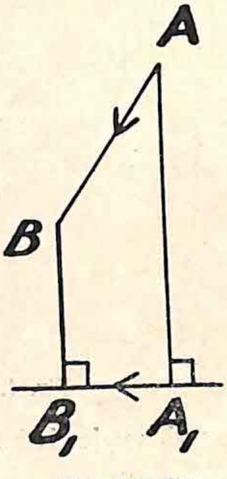


Fig. 1.13 (b)

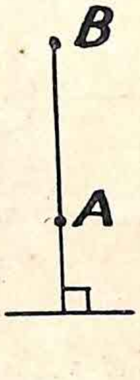


Fig. 1.13 (c)

Definition. The projection (or more precisely the *orthogonal projection*) of a vector $\overrightarrow{AB}$ on a line L is the vector $\overrightarrow{A_1B_1}$ where A_1 and B_1 are the feet of the perpendiculars upon L from A and B respectively (figure 1.12). In case the line L is directed (i.e. positive and negative directions are defined on it), the measure of projection is positive (Fig. 1.13(a)) or negative (Fig. 1.13(b)) according as $\overrightarrow{A_1\,B_1}$ is in the direction of L or in the opposite direction (the unit of measure being in the direction of L); it is zero when $\overrightarrow{AB}$ is perpendicular to L (Fig. 1.13(c)). In the last case the projection is looked upon as the *zero* vector. The measure of the projection is very often referred to as the projection and written $A_1 B_1$ (which may be a positive or negative quantity or zero).

Let $\vec{F}$ be a vector in the co-ordinate plane. We denote its length (i.e. magnitude) by $|\vec{F}|$. We denote by F_x and F_y its projections on the x- and y-axes respectively and their respective magnitudes by $|F_x|$ and $|F_y|$ so that $|F_x| = |\vec{F_x}|$ and $|F_y| = |\vec{F_y}|$.

We have the following theorem :

Theorem. The square of the length of a vector is the sum of the squares of the lengths of its projections on the co-ordinate axes.

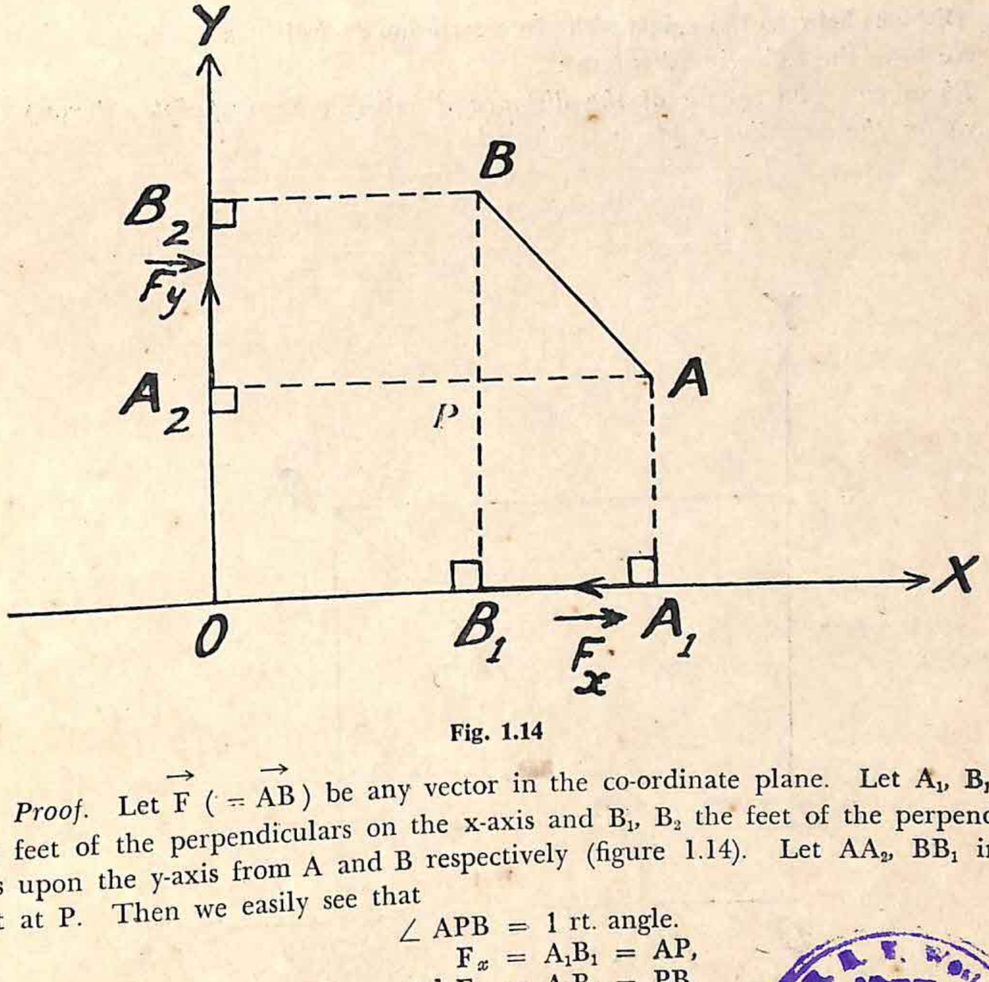


Fig. 1.14

Proof. Let $\vec{F}$ ($= \vec{AB}$) be any vector in the co-ordinate plane. Let A_1, B_1 be the feet of the perpendiculars on the x-axis and B_1, B_2 the feet of the perpendiculars upon the y-axis from A and B respectively (figure 1.14). Let AA_2, BB_1 intersect at P. Then we easily see that

$$\angle APB = 1 \text{ rt. angle.}$$
$$F_x = A_1B_1 = AP,$$
$$\text{and } F_y = A_2B_2 = PB.$$

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Now in the right angled triangle APB,

$$AB^2 = AP^2 + PB^2,$$
$$= (A_1B_1)^2 + (A_2B_2)^2,$$

or $$\boxed{|F|^2 = |F_x|^2 + |F_y|^2}$$

which proves the theorem.

Note. If the vector $\overrightarrow{F}$ is parallel to one of the co-ordinate axes, then the triangle considered in the proof reduces to a segement, but the proof still remains valid because then one of $\overrightarrow{A_1B_1}$ or $\overrightarrow{A_2B_2}$ reduces to a zero vector and the other is equal to $\overrightarrow{AB}$, so that

$$(AB)^2 = (A_1B_1)^2 + (A_2B_2)^2$$

is still true.

1.9. Distance between two points with given cartesian co-ordinates

We have the following theorem :

Theorem. The square of the distance d between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ on the co-ordinate plane is given by

$$\boxed{d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2}$$

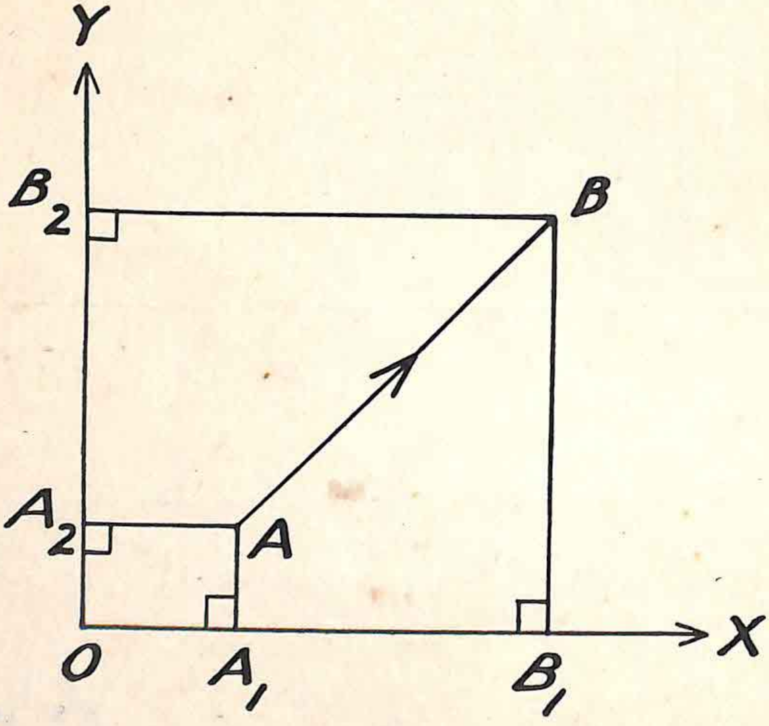

Fig. 1.15

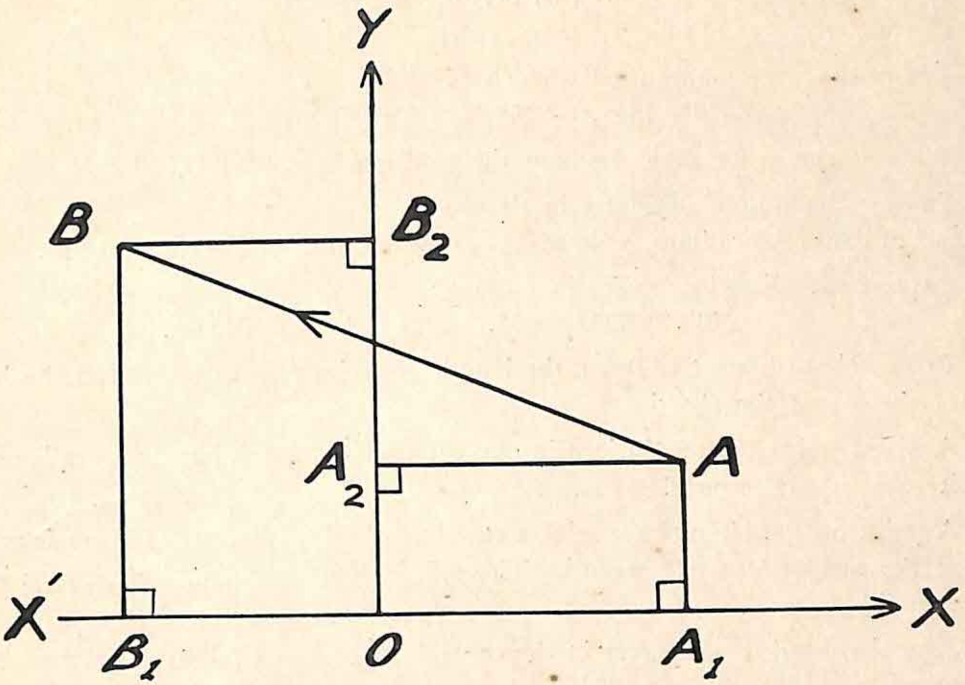

Fig. 1.16

Proof. We consider a vector $\overrightarrow{AB}$ in the co-ordinate plane (figures 1.15 and 1.16). Let A_1B_1, A_2B_2, be the projections of AB on the x-and y-axes respectively. Then by the rule of addition of vectors (directed segments)

$$\overrightarrow{OA_1} + \overrightarrow{A_1B_1} = \overrightarrow{OB_1}, \text{ whence } \overrightarrow{A_1B_1} = \overrightarrow{OB_1} - \overrightarrow{OA_1}, \text{ in both the figures.}$$

As $OA_1 = x$, and $OB_1 = x_2$, the projection of $\overrightarrow{AB}$ on x-axis $= A_1B_1 = x_2 - x_1$.

Similarly the projection of $\overrightarrow{AB}$ on y-axis $= y_2 - y_1$.

Hence by the theorem of Pythagoras.

$$d^2 = (AB)^2 = (A_1B_1)^2 + (A_2B_2)^2,$$

$$\text{or } d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2,$$

which proves the theorem.

Note. The above formula is true for arbitrary disposition of the points A and B. It can be seen by drawing figures for dispositions other than those given in figures 1.14 and 1.15, that the proof remains valid word for word.

ASSORTED EXERCISES

1. Reduce to degrees, minutes and seconds:

(i) 12°·26. (ii) 37°·62. (iii) 42°·15.

2. Write in degrees correct to two decimal places:

(i) $19° \, 17'21''$. (ii) $194'21''$. (iii) $172'41''$.

3. Determine the quadrants in which the following angles lie :

$-36°$, $154°$, $-225°$, $-315°$, $675°$.

4. *What is the acute angle* between the hands of a watch at 3-30 P.M. ?

5. Express the following angles in degrees :

$\pi/3$ radians, $\pi/6$ radians, $\pi/4$ radians, $\pi/20$ radians, 2^c, 3^c, 5^c, 0^c, 30, 10^c.

6. Express the following angles in radians :

$240°$, $60°$, $45°$, $30°$, $18°$, $104°36'$.

7. In a circle of diameter 20 cm, the length of a chord is 10 cm. Find the length of the arc of the chord.

8. A wheel makes 180 revolutions in one minute. Through how many radians does it turn in one second ?

9. A point on the circumference of a rotating wheel of diameter 100 cm is moving at the rate of 50 cm a second. Through how many radians does it turn in one second ?

10. Find the number of degrees in the central angle of a circle of diameter 300 cm by an arc of 22 cm ($\pi = 22/7$).

11. A wheel rotates making 20 revolutions per second. If the radius of the wheel is 50 cm, what linear distance does a point of its rim traverse in three minutes?

12. If in two circles, an arc of the same length subtends angles $60°$ and $75°$ at the centre, find the ratio of their radii.

13. A circular race track is 2 km in circumference. If a man is running on the track at the rate of 8 metres per second what is his angular velocity in degrees and in radians ?

14. A belt moving 15 metres per second passes over a pulley one metre in diameter. What is the angular velocity of the pulley in radians per second ?

15. The equatorial radius of the earth is 6371 km. Find the length of the circumference of the equator ($\pi = 355/113$).

Chapter 2

Trigonometric Functions of Angles

2.1. Definition of Trigonometric Ratios of Arbitrary Angles

Let θ be any angle in standard position and let P(x,y) be any point different from O on its terminal ray OC. Let r denote the length OP. The ratios x/r and y/r are extremely important and useful.

We shall constantly study these ratios during our study of trigonometry. In fact, one may naively say that trigonometry is nothing else but the study of these two ratios (and of certain ratios derived from these ratios). The importance of these ratios is partly due to the following theorem which states that these ratios are independent of the position of P on the terminal ray.

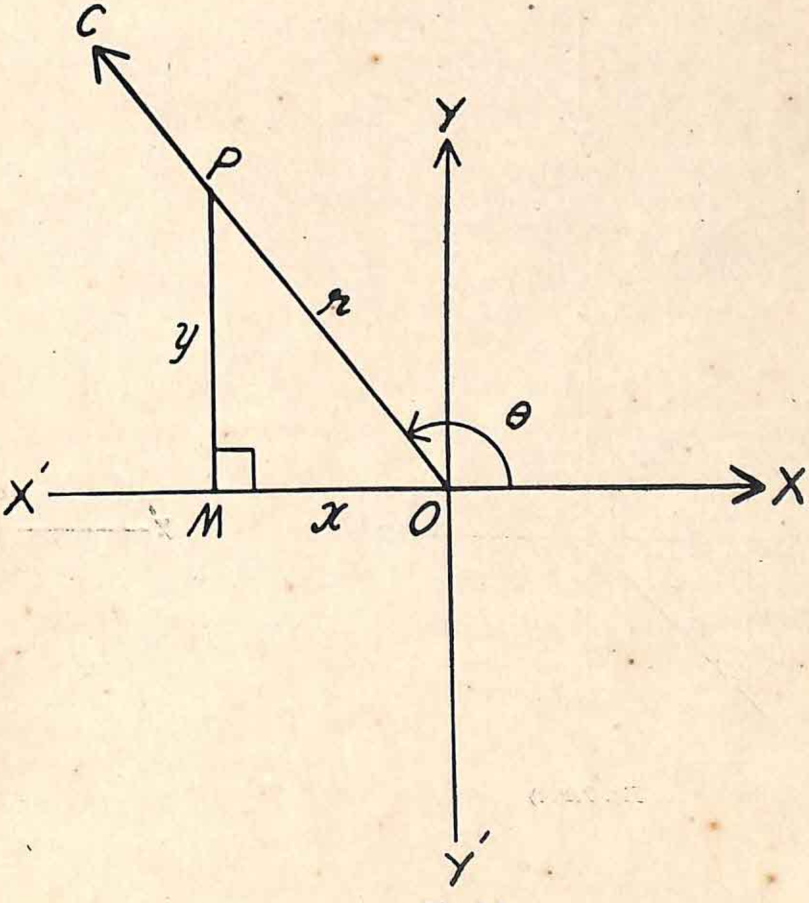

Fig. 2.1

Theorem 2.1. The ratios x/r and y/r are independent of the position of P on the terminal ray.

Proof. We shall prove the result only for the ratio x/r and leave the proof for y/r as an exercise for the reader.

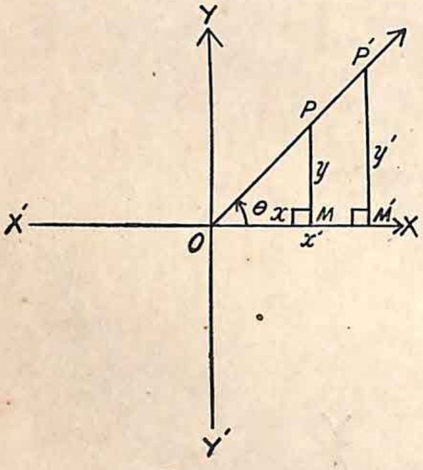

Fig. 2.2 (a)

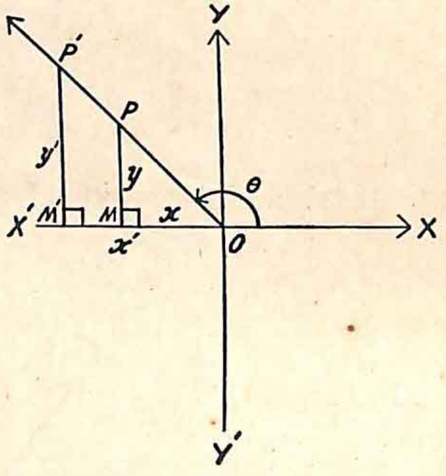

Fig. 2.2 (b)

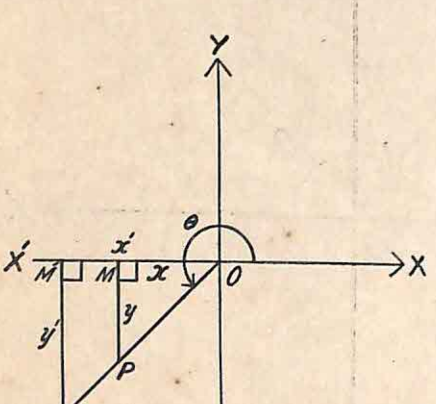

Fig. 2.2 (c)

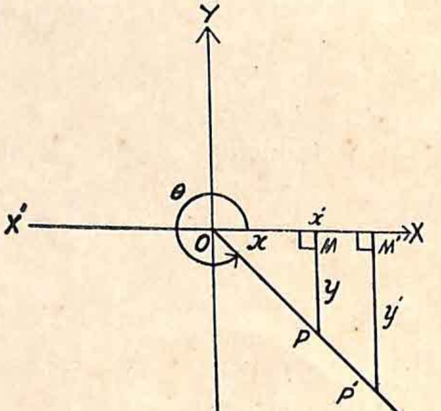

Fig. 2.2 (d)

Case (i). Let us first consider the case in which the terminal ray OC does not coincide with one of the co-ordinate axes. In figures 2.2 (a)—(d), let $P(x,y)$ and $P'(x',y')$ be two different points distinct from the origin on the ray OC. Drawing perpendiculars PM, P'M', to the x-axis, we get similar triangles OPM, OP'M'. The lengths of OM, OM' are $|x|$, $|x'|$ respectively and those of OP, OP' are r, r' respectively. From the similar triangles OPM, OP'M' we have

$$\frac{OM}{OP} = \frac{OM'}{OP'}$$

i.e., $|x|\ r = |x'|\ r'$ (the ratios of the corresponding sides being equal). Thus the ratios x/r and x'/r' are equal in magnitude. Also, since P and P' lie in the same quadrant, therefore x and x' have the same sign. Hence

$$x/r = x'/r'.$$

Case (ii). Let us now consider the case when the terminal ray coincides with one of the co-ordinate semi-axes (Fig. 2.3). We shall discuss the case where the ray OC coincides with OX (and leave the proof for the other cases as exercises for the reader). In this case, for all positions of P, $x/r = 1$ so that x/r is independent of the position of P.

Corollary. The ratio y/x $(x \neq 0)$ is independent of the position of P on the ray.

Proof. Since $x \neq 0$, therfore $x/r \neq 0$.
Also, since by the above theorem y/r and x/r are independent of the position of P on the ray, therefore the ratio $\frac{y/r}{x/r}$ is also independent of the position of P on the ray. Hence the result.

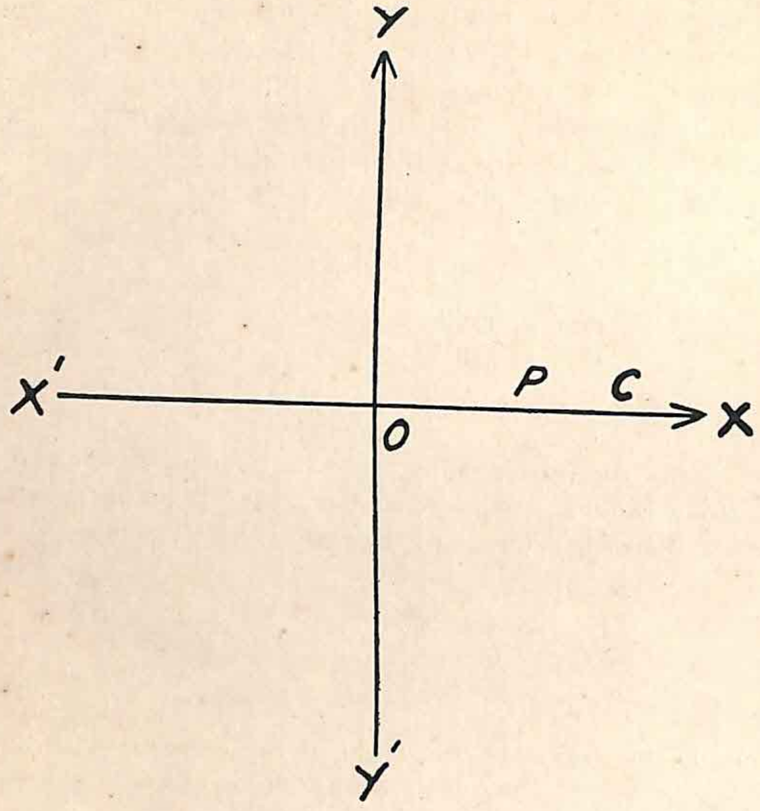

Fig. 2.3

Definition. Let θ be any given angle in standard position, let OC be its terminal ray and let $P(x,y)$ be any point on OC different from O (See figure 2.1).

We define cosine, sine, tangent, cotangent, cosecant and secant of θ (written as $\cos\theta$, $\sin\theta$, $\tan\theta$, $\cot\theta$, $\csc\theta$, $\sec\theta$ respectively) in the following manner:

$\cos\theta = x/r,$

$\sin\theta = y/r,$

$\tan\theta = y/x, \quad (x \neq 0)$

$\cot\theta = x/y, \quad (y \neq 0)$

$\csc\theta = r/y, \quad (y \neq 0)$

$\sec\theta = r/x, \quad (x \neq 0).$

In view of theorem 2.1, the values of these ratios do not depend upon the position of P and these ratios are thus uniquely defined. The ratios $\tan\theta = y/x$ and $\sec\theta = r/x$ are defined if $x \neq 0$. If $x = 0$, then neither of these ratios is defined. Consider, for example, $\theta = \pi/2$. In this case the terminal ray coincides with OY, and consequently the ratios $\tan\theta$ and $\sec\theta$ do not exist when $\theta = \pi/2$.

Similarly, if $y = 0$, then neither of the ratios $\cot\theta = x/y$ and $\csc\theta = r/y$ is defined. Consider for example, $\theta = \pi$. In this case the terminal ray coincides with OX or OX′ so that $y = 0$ and the ratios x/y and r/y are not defined, and consequently the ratios $\cot\theta$ and $\csc\theta$ do not exist when $\theta = \pi$.

Example. The terminal ray of an angle θ in standard position passes through the point $(-4,-3)$. Draw the reference triangle, make the angle θ (with a curved arrow) and write down all the six trigonometric ratios of θ.

Solution. In the right-angled triangle OMP (Fig. 2·4).

$$\begin{aligned} r^2 &= x^2 + y^2, \\ &= (-4)^2 + (-3)^2, \\ &= 25. \end{aligned}$$

Since r cannot be negative, therefore $r=5$.

The trigonometric ratios of θ are, therefore

$$\begin{aligned} \cos\theta &= x/r = -4/5, & \sin\theta &= y/r = -3/5, \\ \tan\theta &= y/x = 3/4, & \cot\theta &= x/y = 4/3, \\ \csc\theta &= r/y = -5/3, & \sec\theta &= r/x = -5/4. \end{aligned}$$

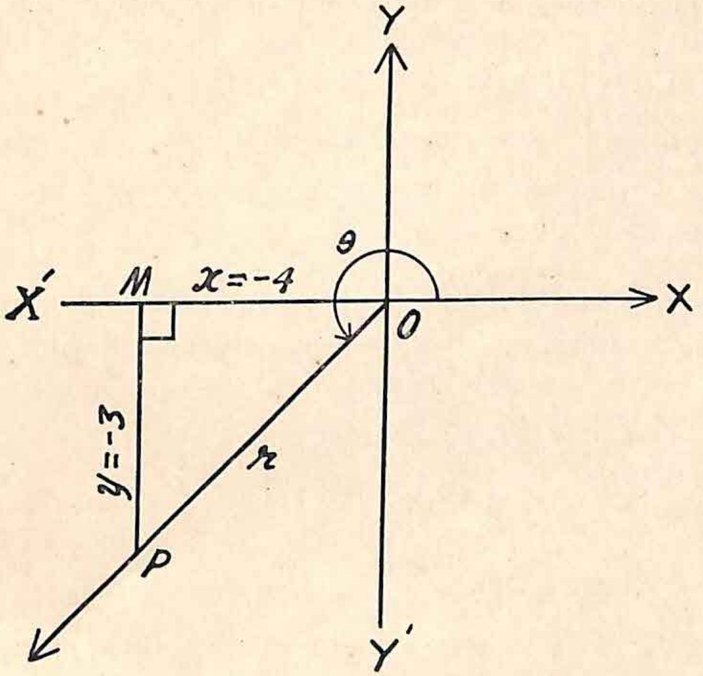

Fig. 2.4

A convention about notation. The trigonometric ratios of an angle have been defined above. Since angles are measured either in degrees or in radians, therefore we adopt the convention that whenever we write $\cos\alpha^\circ$, we mean cosine of the angle whose measure in degrees is α (and similarly for the other ratios), and whenever we write $\cos\beta$ we mean the cosine of the angle whose measure in radians is β (and similarly for other trigonometric ratios).

Some consequences of the definitions of trigonometric ratios. One important consequence of the definitions of the trigonometric ratios is that the values of these ratios are the same for all coterminal angles. For example,

(i) $\cos 380^\circ = \cos(360^\circ + 20^\circ) = \cos 20^\circ$;

(ii) $\sin 460^\circ = \sin(360^\circ + 100^\circ) = \sin 100^\circ$;

(iii) $\tan 800^\circ = \tan(720^\circ + 80^\circ) = \tan 80^\circ$;

(iv) $\cot\left(13\pi/3\right) = \cot(4\pi + \pi/3) = \cot \pi/3$;

(v) $\csc 2\pi$ is not defined because 2π is coterminal with 0 and $\csc 0$ is not defined.

Another important consequence is that whatever θ may be, $\cos\theta \leqslant |1|$ $|\sin\theta| \leqslant 1$, $|\sec\theta| \leqslant 1$, $|\csc\theta| \leqslant 1$. These follow at once from the fact that (see figure 2.1) since $x^2+y^2=r^2$, therefore, $x^2 \leqslant r^2 \wedge y^2 \leqslant r^2$, i.e., $|x| \leqslant r \wedge |y| \leqslant r$ for all positions of P.

Example. Does there exist any value of a for which $2\cos\theta = a+\dfrac{1}{a}$ is a possible equation?

Solution. Since $|\cos\theta| \leqslant |$ for all values of θ, therefore the given equation will be a possible equation provided

$$|1/2\,(a + 1/a)| \leqslant 1.$$

$$\left|\frac{1}{2}\left(a + \frac{1}{a}\right)\right| \leqslant 1 \Longleftrightarrow \left|\frac{a^2+1}{2a}\right| \leqslant 1,$$

$$\Longleftrightarrow (a^2+1)^2 \leqslant 4a^2,$$

$$\Longleftrightarrow (a^2-1)^2 \leqslant 0,$$

$\Longleftrightarrow (a^2-1)^2=0$, since squares of real numbers are always non-negative.

$\Longleftrightarrow a = + 1.$

Therefore the only values of a for which the given relation is a possible equation, are ± 1.

EXERCISES

1. Prove that in theorem 2.1 the radio y/r is independent of the position of the point P on the terminal side.
2. The terminal ray of an angle θ in standard position passes through the point

$(5, -12)$. Draw the reference triangle, mark the angle θ, and write down all the trigonometric ratios of θ.

3. Express each of the following trigonometric ratios in terms of that of an angle lying between 0° and 360° :
(i) tan 1130°; (ii) sin 417°; (iii) co 819°; (iv) cot 633°; (v) sec 1378°; (vi) csc 542°.

4. Is the equation $\sin\theta = a + 1/a$ possible for any real value of a?

5. Does there exist any value of a for which $2\sin\theta = a + 1/a$ is a possible equation ?

6. Is $2\cos^2\theta + \cos\theta - 6 = 0$ a possible equation ?

2.2. The Trigonometric Functions

To every angle θ, there corresponds a ratio (which is a real number) $\cos\theta$. The set $\{(\theta, \cos\theta) : \theta \in A\}$, where A is the set of all angles, therefore defines a function of θ. This function is called the cosine function. Similarly $\{(\theta, \sin\theta) : \theta \in A\}$ is the sine function. The domain of each of these functions is the set A of all angles. We shall later on see that the range of each of these functions is the closed interval $[-1, 1]$.

Again, to every angle $\theta, \theta \neq k\pi + \pi/_2$, where k is any integer, there correspond two ratios $\tan\theta$ and $\sec\theta$. Therefore we have two functions

$$\tan = \{(\theta, \tan\theta) : \theta \in A^*\},$$

and

$$\sec = \{(\theta, \sec\theta) : \theta \in A^*\},$$

where A^* is the set of all angles which are not of the form $k\pi + \pi/_2$ for any integer k. These functions are called the tangent and the secant functions respectively. The domain of each of these functions is A^*. We shall later on see that the range of the tangent function is the set **R** of all real numbers and the range of the secant function is the set $\{x : x \in \mathbf{R} \wedge |x| \geqslant 1\}$ of all real numbers which are numerically greater than or equal to unity.

Finally, to every angle $\theta, \theta \neq p\pi$, where p is any integer, there correspond two ratios $\cot\theta$ and $\csc\theta$. Therefore we have two functions

$$\cot = \{(\theta, \cot\theta) : \theta \in A^{**}\},$$

and

$$\csc = \{(\theta, \csc\theta) : \theta \in A^{**}\},$$

where A^{**} is the set of all angles which are not of the form $p\pi$ for any integer p. These functions are called the cotangent and the cosecant functions respectively. The domain of each of these functions is A^{**}. We shall later on see that the range of the cotangent function is the set of all real numbers, and that of the cosecant is the set $\{x : x \in \mathrm{R} \wedge |x| \leqslant 1\}$.

The six functions defined above are called the *primary* (or *basic*) *trigonometric functions* or *circular functions* or simply the *trigonometric functions*. We shall refer to them by their last mentioned name.

Caution. W should be very careful about our use of cos and cos θ (and similarly about sin and sin θ etc.). Cos is a function, and as such it is a collection of ordered pairs, one ordered pair corresponding to each angle. Cos θ is a trigonometric ratio of the angle θ and as such it is a real number; it is simply the value of the cosine function for the angle θ.

Remarks. 1. As the values of the trigonometric functions are independent of the length of the radius vector $\overrightarrow{OP}$, it is possible to consider this radius vector to be of unit length. Then the point P (figure 2.5) is on the unit circle (circle with centre O and radius unity) and the co-ordinates x and y of P are given by

$$x = \cos\theta\,,\ y = \sin\theta.$$

2. θ In figure 2.5, if is the radian measure of the angle AOP, then θ is also the measure of the arc AP, where A is the point (1, 0). We may, therefore, also regard the trigonometric functions cosine and sine as functions of the arc length (measured from a fixed point up to a variable point) of the unit circle. The other trigonometric functions can also be then defined in terms of this arc length.

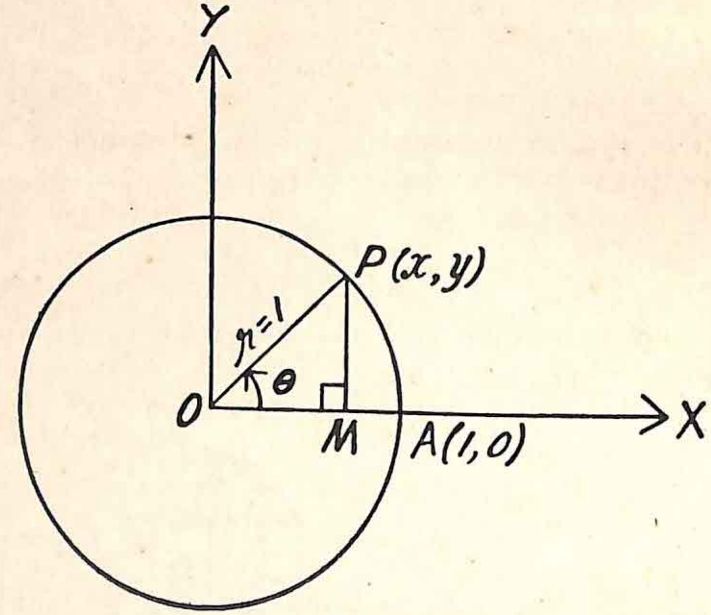

Fig. 2.5

2.3 Signs of the Trigonometric Functions.

The trigonometric functions of an angle are real numbers. Given any angle θ, the value of a trigonometric function of θ may be positive, zero, or negative. While it may need some calculations to determine (even approximately) the values of the trigonometric functions, the signs of the values can be found out just by having a look at the angle (and seeing as to in which quadrant does the terminal ray lie).

Since $\sin\theta = y/r$ and $\csc\theta = r/y$, and since r is always positive, therefore the functions sine and cosecant have the sign of y. Hence the sine and the cosecant of an angle are positive if the angle is in quadrant I or quadrant II, and negative if the angle is in quadrant III or quadrant IV.

Again, since $\cos\theta = x/r$ and $\sec\theta = r/x$, and since r is always positive, therefore the functions cosine and secant have the sign of x. Hence the cosine and the secant of an angle are positive if x is positive, i.e., if the angle is in quadrant I or quadrant IV, and negative if x is negative, i.e., if the angle is in quadrant II or quadrant III.

Finally, since $\tan\theta = y/x$ and $\cot\theta = x/y$, therefore the functions tangent and cotangent are positive if x and y have the same sign, i.e., if the angle is in quadrant I or quadrant III, and negative if x and y have opposite signs, i.e., if the angle is in quadrant II or quadrant IV. The quadrantal angles will be considered separately in the next section.

The above observations can be diagramatically summarized as follows :

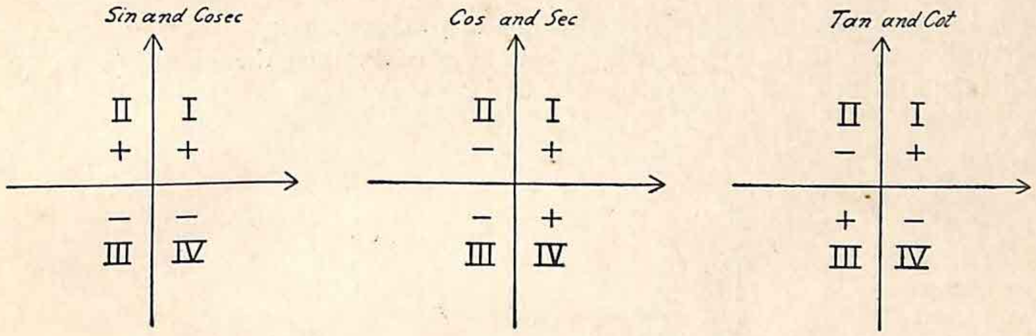

Fig. 2.6

EXERCISES

1. In which quadrant must θ be in order to satisfy each of the following statements?

(i) $\operatorname{Sin}\theta > 0 \wedge \cos\theta > 0$. (ii) $\cos\theta > 0 \wedge \tan\theta < 0$.
(iii) $\sin\theta > 0 \wedge \cot\theta > 0$. (iv) $\sec\theta < 0 \wedge \sin\theta > 0$.
(v) $\csc\theta > 0 \wedge \cot\theta < 0$. (vi) $\tan\theta > 0 \wedge \sin\theta < 0$.
(vii) $\tan\theta < 0 \wedge \csc\theta < 0$. (viii) $\cot\theta > 0 \wedge \cos\theta < 0$.

2. Which of the following statements are true ?
   (i) $\cos 225°$ is a negative number.
   (ii) $\tan\left(\frac{5\pi}{3}\right)$ is a positive number.
   (iii) $\cos 325°$ is a positive number.
3. In which quadrants may θ be in order to satisfy the following statements?
   (i) $\sin\theta < 0$.
   (ii) $\cos\theta > 0$.
   (iii) $\tan\theta > 0$.
   (iv) $\cot\theta < 0$.
   (v) $\sec\theta < 0$.
   (vi) $\csc\theta > 0$.

2.4. Values of Trigonometric Functions of Quadrantal Angles

An angle which in standard position has its terminal ray along one of the co-ordinate axes is called a *quadrantal angle*. Angles of 0°, 90°, 180°, 270° and their coterminal angle are all quadrantal angles. (In fact, these are the only quadrantal angles.)

We propose to find the values of the trigonometric functions of quadrantal angles.

Let us consider the unit circle, so that the end P of the radius vector moves on its circumference. In the case of quadrantal angles, one of the co-ordinates of points on the terminal ray is equal to zero. If the terminal ray lies along the x-axis, then $y = 0$; if the terminal ray lies along the y-axis, then $x = 0$. (This means, geometrically, that if (x, y) are the co-ordinates of P, then the points $(x, 0)$, $(0, 0)$ and (x, y) do not form the vertices of a right-angled triangle. However, the relation $x^2 + y^2 = 1$ holds.)

(i) $\theta = 0°$.

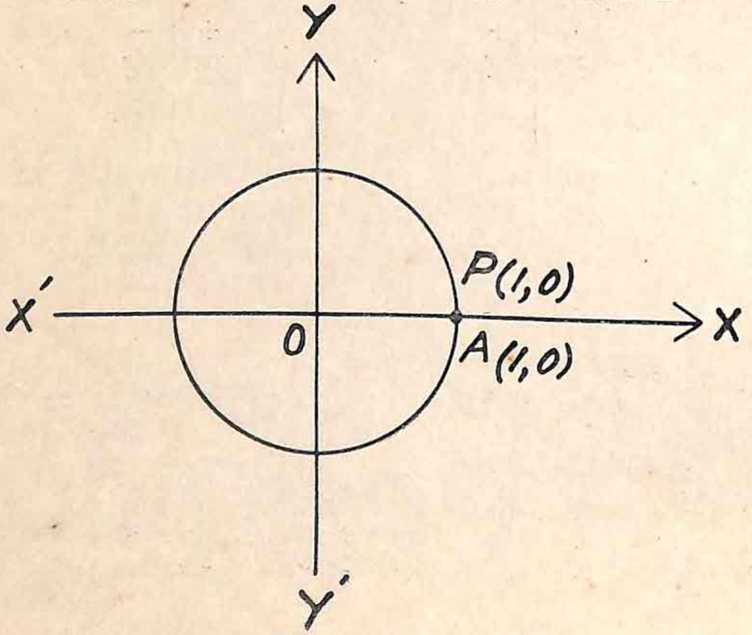

Fig 2.7 (a)

For this angle θ, the end P of the radius vector has co-ordinates $x=1$, $y=0$. Hence $\sin 0°=0$, $\cos 0°=1$, $\tan 0°=0$, $\sec 0°=1$, $\cot 0°$ and $\csc 0°$ do not exist.

(In fact, if we were to try to find out cot 0° and csc 0° in the usual manner, we would see that zero appears in the denominator. Since division by zero is not defined, therefore it follows that an angle of 0° has neither a cotangent nor a cosecant. In mathematical language we may also express this by saying that cot 0° and csc 0° are not defined.

(ii) $\theta = 90°$.

For this angle θ, the end P of the radius vector has co-ordinates $x = 0$ and $y = 1$.

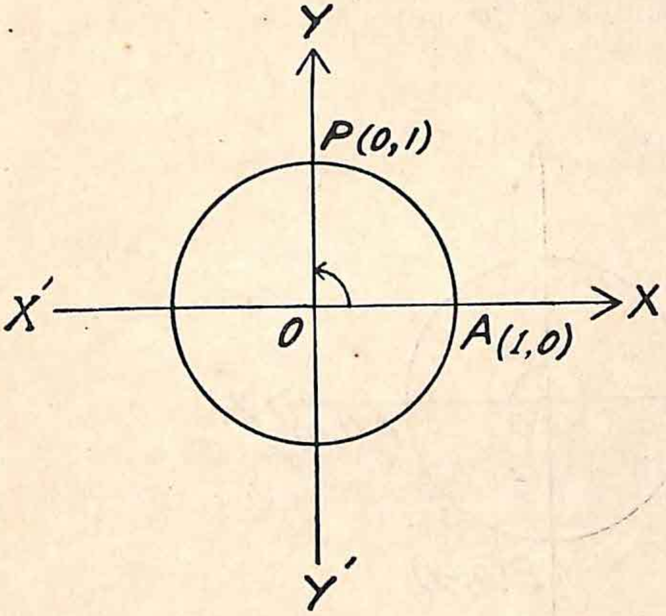

Fig. 2.7 (b)

Hence

$\sin 90° = 1$, $\cos 90° = 0$,
$\cot 90° = 0$, $\csc 90° = 1$,
tan 90° and sec 90° are not defined.

(iii) $\theta = 180°$.

For this angle θ, the end P of the radius vector has co-ordinates $x = -1$, $y = 0$.

Hence

$\sin 180° = 0$, $\cos 180° = -1$,
$\tan 180° = 0$, $\sec 180° = -1$.
cot 180° and csc 180° are not defined.

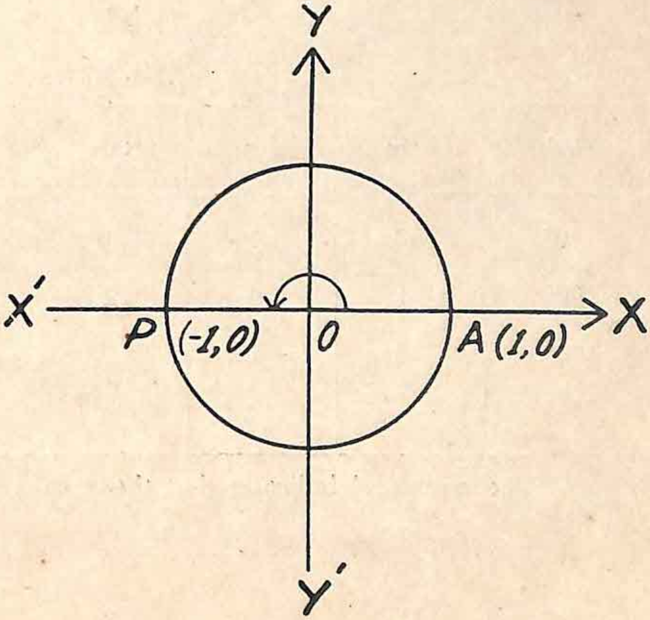

Fig. 2.7 (c)

(iv) $\theta = 270°$.

For this angle θ, the end P of the radius vector has co-ordinates $x = 0$, $y = -1$.

Hence

$\sin 270° = -1$, $\cos 270° = 0$,

$\cot 270° = 0$, $\csc 270° = -1$.

$\tan 270°$ and $\sec 270°$ are not defined.

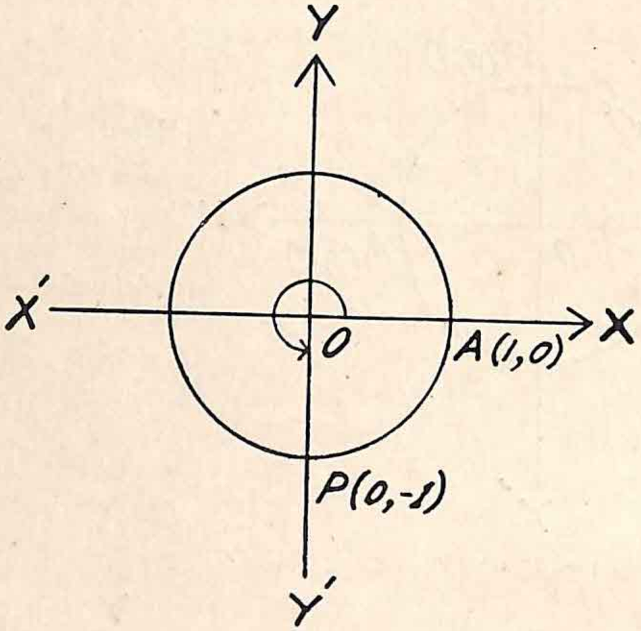

Fig. 2.7 (d)

Angle	sin	cos	tan	cot	sec	csc
0°	0	1	0	×	1	×
90°	1	0	×	0	×	1
180°	0	−1	0	×	−1	×
270°	−1	0	×	0	×	−1

Remark. The entry '×' indicates that the trigonometric ratio in question is not defined.

EXERCISES

1. Fill up the blanks :

$\sin \frac{\pi}{2} =$ ………………………… $\cos \frac{3\pi}{2} =$ …………………………

$\tan 0 =$ ………………………… $\cot \frac{\pi}{2} =$ …………………………

$\sec \pi =$ ………………………… $\sec \pi =$ …………………………

$\csc \frac{3\pi}{2} =$ ………………………… $\csc \frac{\pi}{2} =$ …………………………

2. Find the value of each of the following expressions :

(i) $\sin^2 \frac{\pi}{2} + \cos^2 \frac{\pi}{2}$. (ii) $1 + \cot^2 \frac{3\pi}{2}$.

(iii) $\sin 270° \cos 180° + \sin 180° \cos 270°$.

(iv) $\dfrac{\cot 90° - \cot 270°}{1 + \cot 90° \cot 270°}$.

3. Name two trigonometric functions of θ which are not defined when $\theta = 270°$.

4. Why are $\tan \frac{\pi}{2}$ and $\csc \pi$ not defined ?

2.5. Trigonometric Functions of Positive Acute Angles

Values of trigonometric functions of acute angles can be easily computed by

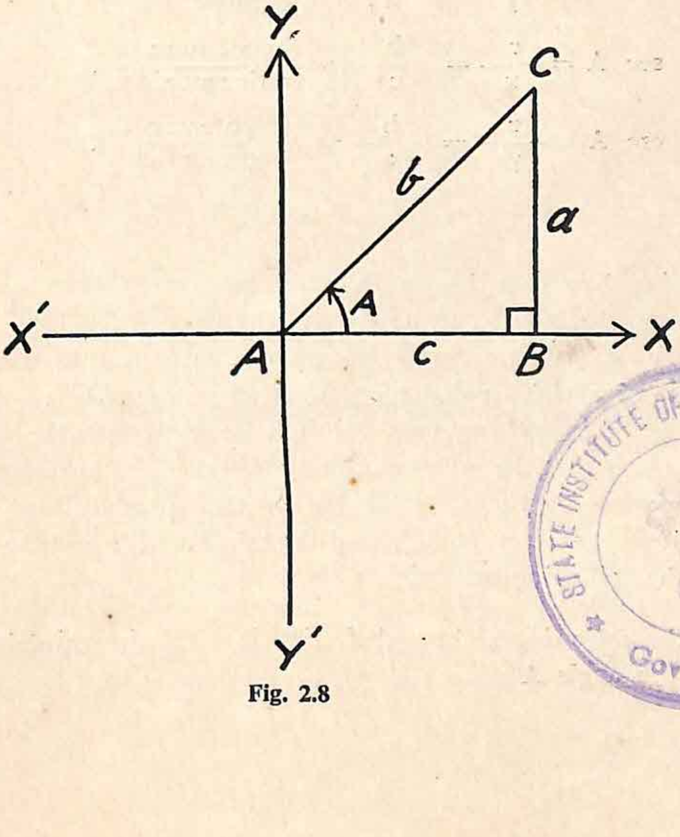

Fig. 2.8

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considering right angled triangles without introducing a co-ordinate system. In figure 2.8, we have a right-angled triangle ABC, right-angled at B. The side AB is called the *adjacent side* and the side BC is called the *opposite side* of the acute angle A. AC is the hypotenuse. As usual, we denote the lengths of the BC, CA, and AB by a, b, and c respectively.

Let us choose A as the origin of co-ordinates, take the axis of x along AB, and the axis of y along the straight line perpendicular to AB through the point A. The angle BAC is then in the standard position, with AC as the terminal ray. The co-ordinates of C are (c, a), so that $x = c$, $y = a$, $r = b$.

$$\sin A = \frac{y}{r} = \frac{a}{b} = \frac{\text{opposite side}}{\text{hypotenuse}},$$

$$\cos A = \frac{x}{r} = \frac{c}{b} = \frac{\text{adjacent side}}{\text{hypotenuse}},$$

$$\tan A = \frac{y}{x} = \frac{a}{c} = \frac{\text{opposite side}}{\text{adjacent side}},$$

$$\cot A = \frac{x}{y} = \frac{c}{a} = \frac{\text{adjacent side}}{\text{opposite side}},$$

$$\sec A = \frac{r}{x} = \frac{b}{c} = \frac{\text{hypotenuse}}{\text{adjacent side}},$$

$$\csc A = \frac{r}{y} = \frac{b}{a} = \frac{\text{hypotenuse}}{\text{opposite side}}.$$

These formulae show that the values of trigonometric functions in the above case depend only on the lengths of the sides of the triangle ABC and are independent of the choice of any co-ordinate system. We may, therefore, compute the values for either acute angle of a right-angled triangle from the lengths of the sides of the triangle. Moreover, it is not necessary for the triangle to be in the standard position (i.e., it is not necessary that the adjacent side be parallel to the printed lines of the page and the opposite side be perpendicular to the printed lines). The essential thing is to pick out the adjacent side and the opposite side and then use the above formulae as in the following example

Example. For the following right-angled triangle, find the values of the trigonometric ratios of the angle A.

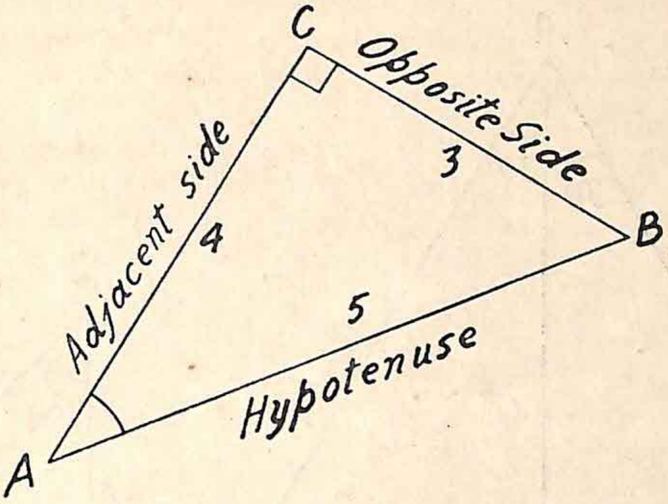

Fig. 2.9

Solution. With respect to the angle A, the side AC is the adjacent side and the side BC is the opposite side.
Therefore,

$$\sin A = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{BC}{AB} = \frac{3}{5},$$
$$\cos A = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{AC}{AB} = \frac{4}{5},$$
$$\tan A = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{BC}{AC} = \frac{3}{4},$$
$$\cot A = \frac{\text{adjacent side}}{\text{opposite side}} = \frac{AC}{BC} = \frac{4}{3},$$
$$\sec A = \frac{\text{hypotenuse}}{\text{adjacent side}} = \frac{AB}{AC} = \frac{5}{4},$$
$$\csc A = \frac{\text{hypotenuse}}{\text{opposite side}} = \frac{AB}{BC} = \frac{5}{3},$$

2.6. Values of Trigonometric Functions of Angles of 30°, 45°, and 60° ($\pi/6$, $\pi/4$ and $\pi/3$)

The values of trigonometric functions of angles of 30°, 45°, and 60° can be found very easily by considering certain right-angled triangles.

To compute the values of trigonometric functions of angles of 30° and 60°, we construct an equilateral triangle ABC with each side of length 2 units and draw AD the bisector of $\angle$ BAC. We can easily prove (by showing that the triangles ADB and ADC are congruent) that BD = DC = 1 unit, AC = AB = 2 units,

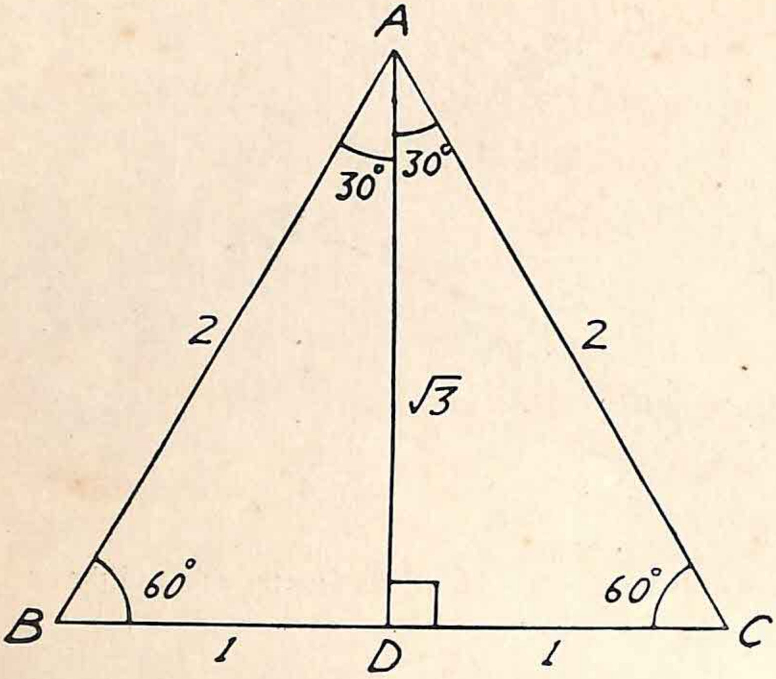

Fig. 2.10

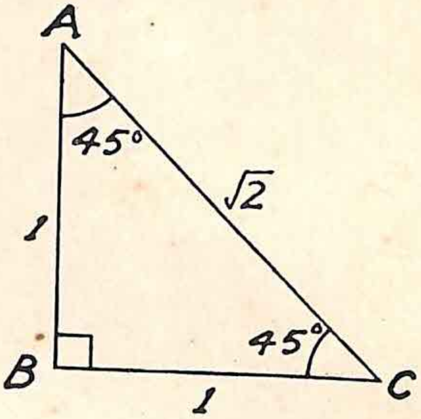

Fig. 2.11

$\angle$ BAD = $\angle$ CAD = 30°, $\angle$ B = $\angle$ C = 60° and $\angle$ ADC = 90°. By applying Pythagoras theorem to the right-angled triangle ADC, we have $AD = \sqrt{(AC^2 - DC^2)} = \sqrt{(2^2 - 1^2)} = \sqrt{3}$ units. By considering the right-angled triangle ADC, we have

$$\sin 30° = \sin \widehat{DAC} = CD/AC = 1/2,$$

$$\cos 30° = \cos \widehat{DAC} = AD/AC = \sqrt{3}/2$$

$$\tan 30° = \tan \widehat{DAC} = CD/AD = 1/\sqrt{3},$$

$$\cot 30° = \cot \widehat{DAC} = AD/CD = \sqrt{3},$$

$$\sec 30° = \sec \widehat{DAC} = AC/AD = 2/\sqrt{3},$$

$$\csc 30° = \csc \widehat{DAC} = AC/CD = 2.$$

Similarly, by considering the angle DCA, we have

$\sin 60° = \sqrt{3}/2$, $\cos 60° = 1/2$, $\tan 60° = \sqrt{3}$,
$\csc 60° = 2/\sqrt{3}$, $\sec 60° = 2$, $\cot 60° = 1/\sqrt{3}$.

To compute the values of trigonometric functions of an angle of 45°, we construct an isosceles triangle ABC (figure 2.11) in which each of the equal sides is equal to 1 unit. By Pythagoras theorem,

$$AC = \sqrt{\{(BA)^2 + (BC)^2\}} = \sqrt{2} \text{ units.}$$

$$\text{Also, } \angle BCA = \angle BAC = 45^\circ.$$

$$\sin 45^\circ = \sin \angle BCA = AB/AC = 1/\sqrt{2},$$
$$\cos 45^\circ = \cos \angle BCA = BC/AC = 1/\sqrt{2},$$
$$\tan 45^\circ = \tan \angle BCA = AB/BC = 1,$$
$$\cot 45^\circ = \cot \angle BCA = BC/AB = 1,$$
$$\sec 45^\circ = \sec \angle BCA = AC/BC = \sqrt{2},$$
$$\csc 45^\circ = \csc \angle BCA = AC/AB = \sqrt{2}.$$

Remarks. 1. Comparing the values of the trigonometric functions of pairs of complementary angles 30°, 60°, 45°, 45° we note that the sine of the one is equal to the cosine of the other, the tangent of the one is equal to the cotangent of the other, and the secant of the one is equal to the cosecant of the other. We shall see in the next chapter that these relations are valid for any pair of complementary angles.

2. The above method for finding the values of trigonometric functions of angles of 30°, 45°, and 60° can be applied to compute the values of the trigonometric functions of an angle in any quadrant if the terminal ray of the angle makes one of these angles (30°, 45°, 60°) with the positive or negative x-axis. The following example will illustrate the method :

Example. Construct an angle of 240° in standard position and find the values of its trigonometric functions.

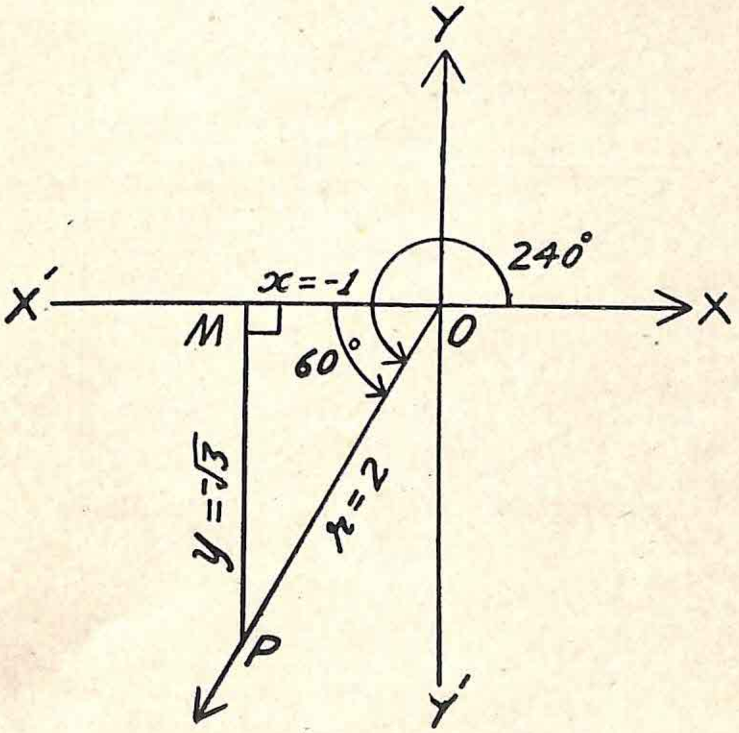

Fig. 2.12

Solution. The terminal ray of the angle of 240° is in the third quadrant and makes an angle of 60° with OX′. Take a point P on the terminal ray, so that OP = 2 units and draw PM perpendicular to OX′. It can be easily shown that OM = 1 unit, MP = $\sqrt{3}$ units. Therefore the co-ordinates of P are $x = -1$, $y = -\sqrt{3}$.

Therefore,

$\sin 240^\circ = y/r = -\sqrt{3}/2$, $\csc 240^\circ = r/y = -2/\sqrt{3}$,
$\cos 240^\circ = x/r = -1/2$, $\sec 240^\circ = r/x = -2$,
$\tan 240^\circ = y/x = \sqrt{3}$, $\cot 240^\circ = x/y = \sqrt{1/3}$.

EXERCISES

1. For each of the following right-angled triangles, write the sine, cosine and tangent of each of the acute angles :

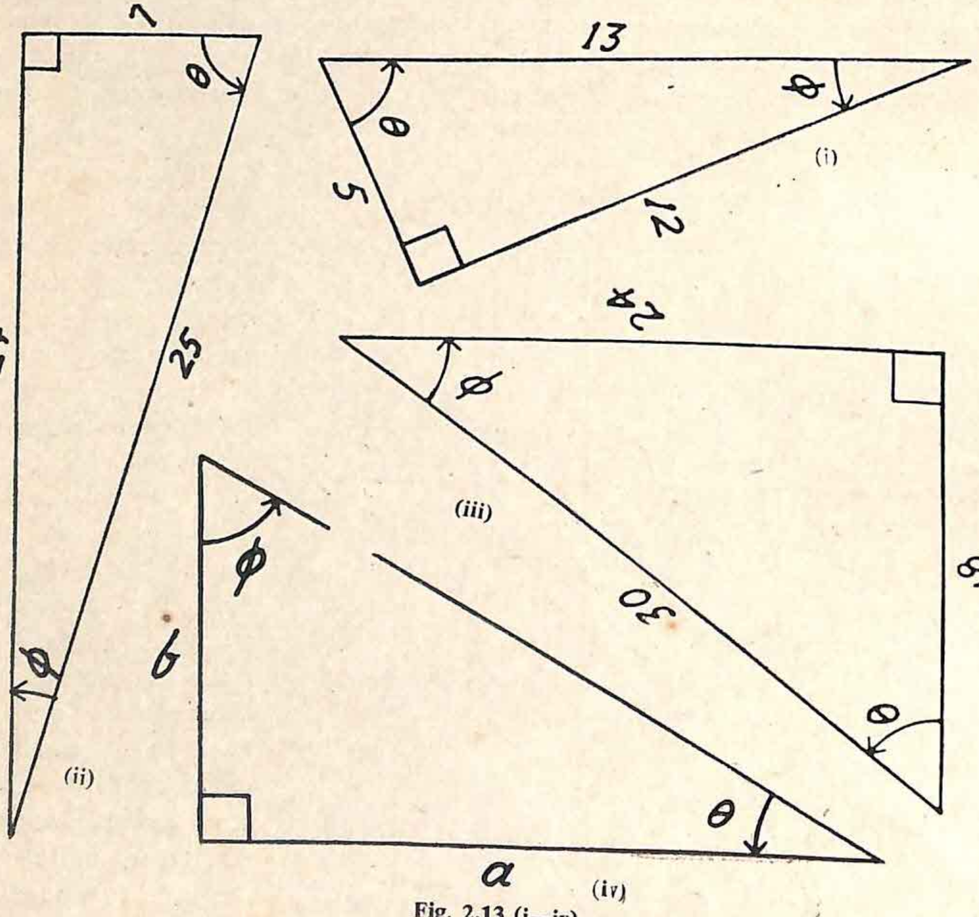

Fig. 2.13 (i—iv)

2. Complete the following table :

Angle	sin	cos	tan	cot	sec	csc
$\pi/6$						
$\pi/4$						
$\pi/3$						

3. Evaluate each of the following expressions :

(i) $\sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$

(ii) $\cos 60^\circ \cos 45^\circ - \sin 60^\circ \sin 45^\circ$.

(iii) $\dfrac{\tan \pi/4 - \tan \pi/6}{1 + \tan \pi/4 \tan \pi/6}$. (iv) $\dfrac{\cot \pi/4 \cot \pi/3 + 1}{\cot \pi/3 - \cot \pi/4}$.

4. Substitute the values of trigonometric functions and show that each of the following statements is true :

(i) $\sec^2 30^\circ + \cos^2 30 = 1$. (ii) $\sec^2 45^\circ = 1 + \tan^2 45^\circ$.

(iii) $\csc^2 \pi/4 = 1 + \cot^2 \pi/4$ (iv) $\sin \pi/3 = 2 \sin \pi/6 \cos \pi/6$.

(v) $\cos 30^\circ = \cos 60^\circ \cos 30^\circ + \sin 60^\circ \sin 30^\circ$.

(vi) $\sin \pi/6 = \sin \pi/3 \cos \pi/6 - \sin \pi/6 \cos \pi/3$.

(vii) $\tan 30^\circ = \dfrac{\tan 60^\circ - \tan 30^\circ}{1 + \tan 60^\circ \tan 30^\circ}$ (viii) $\tan \pi/3 = \dfrac{2 \tan \pi/6}{1 - \tan^2 \pi/6}$.

(ix) $\cot 30^\circ = \dfrac{1 + \cos 60^\circ}{\sin 60^\circ}$ (x) $\sec \pi/3 = \dfrac{1 - \cot^2 \pi/3}{1 + \cot^2 \pi/3}$.

5. Construct an equilateral triangle of each side equal to 1 unit, and use it to write down the values of trigonometric functions of angles π /6 and π3.

6. Write down, with the help of figure 2.14, the values of trigonometric functions of angle of 300°.

7. Construct an angle of $\frac{5\pi}{6}$ in standard position and find the values of its trigonometric functions.

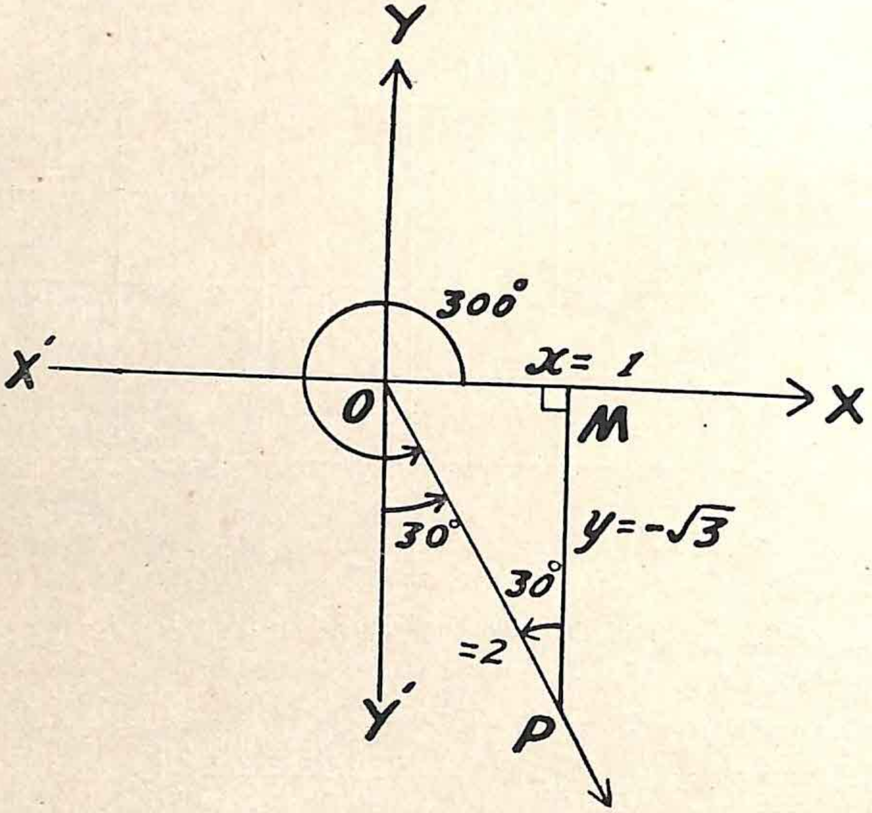

Fig. 2.14

2.7 Trigonometric Identities

An equation that holds true for all those values of the variables for which both sides of the equation are meaningful is called an *identity*. (Those values of the variables for which both sides of an equation are meaningful are called *permissible* values.)

Illustrations.

1. $x^2 - 1 = (x + 1)(x - 1)$ is an identity because it holds true for all values of x.

2. $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$ is an identity because it holds true for all values of x and y.

3. $\frac{1}{x - 1} + \frac{1}{x - 1} = \frac{2x}{x^2 - 1}$ is an identity because it holds true for all permissible values of x, that is, for all those values of x for which the two sides have a meaning. (Both sides of the given equation have meaning for all values of x except $x = 1$ and $x = -1$.)

4. $2x - 4 = x + 3$ is not an identity because it does not hold true for all permissible values of x. Here, both sides of the equation have a meaning for all values of x, but the equation holds true only when $x = 7$.

5. $\frac{1}{x-1}+\frac{2}{x-2}=\frac{3}{x-3}$ is not an identity because it does not hold true for all permissible values of x. Here both sides of the equation have a meaning except when $x = 1, 2$ or 3, but the equation holds true only when $x = 3/2$.

An equation that holds true for some, but not all, of the permissible values is called a *conditional equation*. Illustrations 4—6 above are examples of conditional equations.

The distinction between an identity and a conditional equation can be very aptly put as follows :

$\frac{\text{An identity}}{\text{A conditional equation}}$ is an equation that $\frac{\text{holds true}}{\text{does not hold true}}$ for all permissible values of the variables.

In the present section we propose to consider identities involving trigonometric functions. Such identities are very useful in various fields, especially in physics and engineering sciences.

The fundamental identities. We give below some identities involving the trigonometric functions. These identities follow directly from the definitions of the trigonometric functions and are therefore called *fundamental identities.*

I	$\tan\theta = \sin\theta/\cos\theta$,	$\cot\theta = \cos\theta/\sin\theta$.
II	$\sec\theta = 1/\cos\theta$,	$\csc\theta = 1/\sin\theta$.
III	$\tan\theta = 1/\cot\theta$,	$\cot\theta = 1/\tan\theta$.

The above identities follow at once from the definitions. (How ?).

IV	$\cos^2\theta + \sin^2\theta = 1$.
V	$1 + \tan^2\theta = \sec^2\theta$.
VI	$1 + \cot^2\theta = \csc^2\theta$.

Proof. Let θ be any angle. (See Fig. 2.5.) Let $\overrightarrow{OP}$ be the radius vector of the unit circle making the angle θ with the positive x-axis (i.e., $\overrightarrow{OP}$ is along a ray from 0 making an angle θ with the positive x-axis). Then $x = \cos\theta$, $y = \sin\theta$.

Since the length of $\overrightarrow{OP} = 1$, therefore $x^2 + y^2 = 1$,
or $\cos^2\theta + \sin^2\theta = 1$. ....(IV)

Dividing both sides of IV by $\cos^2\theta$ ($\cos^2\theta \neq 0$), we have,

$$1 + \sin^2\theta/\cos^2\theta = 1/\cos^2\theta\,,$$

$$\text{or } 1 + \tan^2\theta = \sec^2\theta\,, \qquad \text{............(V)}$$

by I and II.

Again, dividing both sides of IV by $\sin^2\theta$ ($\sin^2\theta \neq 0$), we have

$$\cos^2\theta/\sin^2\theta + 1 = 1/\sin^2\theta\,,$$

$$\text{or } \cot^2\theta + 1 = \operatorname{cosec}^2\theta\,, \qquad \text{............(VI)}$$

by I and II.

Remark. In the above lines we have written $\sin^2\theta$ for $(\sin\theta)^2$, $\cos^2\theta$ for $(\cos\theta)^2$, $\tan^2\theta$ for $(\tan\theta)^2$ and so on. In general, we shall write $\sin^m\theta$ for $(\sin\theta)^m$, $\cos^m\theta$ for $(\cos\theta)^m$, $\tan^m\theta$ for $(\tan\theta)^m$ and so on for $m > 0$ but never so for $m \leqslant 0$ (see chapter V).

The above identities are used for establishing more complicated identities, and also for transforming trigonometric expressions. The methods will be best illustrated by examples.

Example. Prove the identity

$$\frac{\sin^2\theta}{1+\cos\theta} = 1 - \cos\theta, \quad 1 + \cos\theta \neq 0.$$

Solution.

First Method

$$\begin{aligned}
\text{L.H.S.} &= \frac{\sin^2\theta}{1+\cos\theta}\,,\\
&= \frac{1-\cos^2\theta}{1+\cos\theta}\,,\\
&= \frac{(1+\cos\theta)(1-\cos\theta)}{1+\cos\theta}\,,\\
&= 1 - \cos\theta, \text{ since } 1+\cos\theta \neq 0,\\
&= \text{R.H.S.}
\end{aligned}$$

Second Method

$$\begin{aligned}
&\text{Let} && \frac{\sin^2\theta}{1+\cos\theta} \neq 1 - \cos\theta,\ 1+\cos\theta \neq 0, \text{ if possible}\\
&\text{Then} && \frac{\sin^2\theta\,(1+\cos\theta)}{1+\cos\theta} \neq (1-\cos\theta)(1+\cos\theta),\\
&\text{or} && \sin^2\theta \neq 1 - \cos^2\theta,\\
&\text{or} && \sin^2\theta + \cos^2\theta \neq 1,
\end{aligned}$$

which is impossible since we know that

$$\sin^2\theta + \cos^2\theta = 1, \text{ for all values of } \theta.$$

The contradiction shows that

$$\frac{\sin^2\theta}{1+\cos\theta} = 1 - \cos\theta,\ 1 + \cos\theta \neq 0.$$

Third Method

From the fundamental identity

$$\cos^2\theta + \sin^2\theta = 1,$$

we have

$$\begin{aligned} & \sin^2\theta = 1 - \cos^2\theta, \\ \text{or} \quad & \sin^2\theta = (1 - \cos\theta)(1 + \cos\theta). \end{aligned}$$

Since $1 + \cos\theta \neq 0$, therefore dividing both sides by $1 + \cos\theta$, we have

$$\frac{\sin^2\theta}{1 + \cos\theta} = 1 - \cos\theta.$$

Hence the identity.

Remarks. 1. The technique employed for the proof in the Second Method is called *proof by contradiction*. This is an extremely important method of proof and is employed very frequently and in very diverse situations. You might have already used this method in proving some theorems in plane geometry. (Can you recall two instances from plane geometry where this method of proof is used ?).

2. Consider the following 'false proof' this method of proof is used ?)

$$\frac{\sin^2\theta}{1 + \cos\theta} = 1 - \cos\theta,\ 1 + \cos\theta \neq 0 \qquad \ldots\ldots\ldots\ldots(1)$$

Multiplying throughout by $1 + \cos\theta$ to clear off the fractions, we have

$$\begin{aligned} & \sin^2\theta = (1 - \cos\theta)(1 + \cos\theta), \\ \text{or} \quad & \sin^2\theta = (1 - \cos\theta)(1 + \cos\theta), \\ \text{or} \quad & \sin^2\theta = (1 - \cos^2\theta), \\ \text{or} \quad & \sin^2\theta = \sin^2\theta, \qquad \ldots\ldots\ldots\ldots(2) \end{aligned}$$

which is true. Hence the identity (1) is true.

Can you guess as to what is wrong with the above mode of reasoning ? Here we started with (1), about the truth of falsity of which we did not know and we arrived at (2) which is known to be true, by a chain of valid operations and arguments. From this it is not possible for us to arrive at the conclusion that (1) must be true. A false hypothesis may and in fact does lead to a correct conclusion. This may be illustrated by the following example :

Is the following equation an identity :

$$\sin\theta - \csc\theta = \cot\theta\ \cos\theta\ ? \qquad \ldots\ldots\ldots\ldots(\text{i})$$

Let us proceed as follows :

Squaring both sides of (i), we have

$$(\sin\theta - \csc\theta)^2 = \cot^2\theta \cos^2\theta,$$

or $$\sin^2\theta + \csc^2\theta - 2 = \cot^2\theta\,(1 - \sin^2\theta),$$

or $$\sin^2\theta + \csc^2\theta - 2 = \cot^2\theta - \cos^2\theta.$$

Transposing all the terms to the left hand side, we have

$$(\sin^2\theta + \cos^2\theta) + (\csc^2\theta - \cot^2\theta) - 2 = 0,$$

or $$1 + 1 - 2 = 0, \text{ since } \sin^2\theta + \cos^2\theta = 1 \text{ and } \operatorname{cosec}^2\theta - \cot^2\theta = 1,$$

or $$0 = 0. \qquad \text{............(ii)}$$

Since (ii) holds true, therefore we may be tempted to conclude that (i) holds true for all permissible values of θ. However, by putting $\theta = \pi/4$ in (i), we may say that (i) is not an identity. In fact, when $\theta = \pi/4$, the left hand side of (i) equals $1/\sqrt{2} - \sqrt{2}$, i.e., $-1/\sqrt{2}$, and the right hand side equals $1/\sqrt{2}$.

Examples. Prove the identity

$$\tan\theta + \cot\theta = \sec\theta \operatorname{cosec}\theta.$$

Solution.

$$\begin{aligned}
\text{L.H.S.} &= \sin\theta/\cos\theta + \cos\theta/\sin\theta,\\
&= \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta},\\
&= \frac{1}{\cos\theta\sin\theta},\\
&= \frac{1}{\cos\theta}\cdot\frac{1}{\sin\theta},\\
&= \sec\theta\operatorname{cosec}\theta,\\
&= \text{R.H.S.}
\end{aligned}$$

Example. Prove the identity

$$2\tan\theta = \frac{\cos\theta}{1 - \sin\theta} - \frac{\cos\theta}{1 + \sin\theta}.$$

Solution. Since the right hand side is more complicated, we proceed by performing the operations indicated on the right hand side.

$$\begin{aligned}
\text{R.H.S.} &= \frac{\cos\theta}{1 - \sin\theta} - \frac{\cos\theta}{1 + \sin\theta},\\
&= \frac{\cos\theta\,(1 + \sin\theta) - \cos\theta\,(1 - \sin\theta)}{(1 - \sin\theta)(1 + \sin\theta)},\\
&= \frac{2\cos\theta\sin\theta}{\cos^2\theta},\\
&= 2\tan\theta,\\
&= \text{L.H.S.}
\end{aligned}$$

Example. Is the following equation an identity ?

$$\cos^3\theta + \sin^3\theta = 1.$$

Solution. The given equation is not an identity, because $\theta = \pi/4$ is a permissible value for which the left hand side equals $(\cos \pi/4)^3 + (\sin \pi/4)^3 = (1/\sqrt{2})^3 + (1/\sqrt{2})^3 = 1/\sqrt{2}$, which is not equal to the right hand side.

Example. Prove the identity

$$\sqrt{\left\{\frac{2}{1+\cos\theta} + \frac{2}{1-\cos\theta}\right\}} = \frac{2}{|\sin\theta|}.$$

Solution.

$$\begin{aligned} \text{L.H.S.} &= \sqrt{\left\{\frac{2}{1+\cos\theta} + \frac{2}{1-\cos\theta}\right\}}, \\ &= \sqrt{\left\{\frac{2(1-\cos\theta) + 2(1+\cos\theta)}{1-\cos^2\theta}\right\}}, \\ &= \sqrt{\left\{\frac{4}{\sin^2\theta}\right\}}, \\ &= \frac{2}{|\sin\theta|}, \\ &= \text{R.H.S.} \end{aligned}$$

Remark. While no definite rules can be laid down for proving (or disproving) an identity, the following suggestions may prove useful :

1. Do not work with the equation as if it were a conditional equation.

2. It generally proves helpful to change the more complicated side of the equation into the form of the other side.

3. If one side involves only one function of the angle, it may prove to be of advantage to express the other side in terms of this function.

4. It is often helpful to change all the functions into sines and cosines.

5. Always keep an eye on the side which is being kept unaltered, for this is the goal which we have to reach. We have to watch constantly as to what transformations will lead us to this goal.

6. Factoring is useful sometimes.

7. Sometimes it is useful to multiply the numerator and denominator by a factor which does not vanish within the domain of the variable.

8. As far as possible, try to avoid introducing radicals.

EXERCISES

1. Prove the following identities :
   - (i) $\cos^4\theta - \sin^4\theta = \cos^2\theta - \sin^2\theta$.
   - (ii) $(\tan\theta + \cot\theta)^2 = \csc^2\theta + \sec^2\theta$.
   - (iii) $\tan^4\theta - \sec^4\theta = 1 - 2\sec^2\theta$.
   - (iv) $(\sin\theta - \tan\theta)(\csc\theta + \cot\theta) = \cos\theta - \sec\theta$.
   - (v) $(1 + \cot\theta - \csc\theta)(1 + \tan\theta + \sec\theta) = 2$.

(vi) $\dfrac{1-\cos\theta}{1+\cos\theta}=(\csc\theta-\cot\theta)^2$.

(vii) $\dfrac{1+\sin\theta}{\cos\theta}=\dfrac{\cos\theta}{1-\sin\theta}$.

(viii) $\dfrac{\cot\theta}{\csc\theta-1}=\dfrac{\csc\theta+1}{\cot\theta}$.

(ix) $\dfrac{1}{\tan\theta-\sec\theta}+\dfrac{1}{\tan\theta+\sec\theta}=-2\tan\theta$.

(x) $\dfrac{\sec\theta}{1+\cos\theta}=\dfrac{\sec\theta-1}{\sin^2\theta}$.

2. Is the equation in the following problems an identity or a conditional equation ?

(i) $(\tan\theta+\sec\theta)^2=\dfrac{1+\sin\theta}{1-\sin\theta}$.

(ii) $\dfrac{\sin\theta\tan\theta}{1-\cos\theta}=1+\sec^2\theta$.

(iii) $\sec\theta-\tan\theta=\sqrt{\left\{\dfrac{1-\sin\theta}{1+\sin\theta}\right\}}$.

(iv) $\dfrac{1+\cos\theta+\sin\theta}{1+\cos\theta-\sin\theta}=\dfrac{1+\sin\theta}{\cos\theta}$.

(v) $\sec\theta\,(1-\sin\theta)=\dfrac{1}{\sec\theta+\tan\theta}$.

3. Verify the truth of the following statements, when θ lies in the first quadrant :

(i) $\sqrt{\left\{\dfrac{1+\cos\theta}{1-\cos\theta}\right\}}=\csc\theta+\cot\theta$.

(ii) $\sqrt{\left\{\dfrac{\sec\theta-1}{\sec\theta+1}\right\}}=\dfrac{\sin\theta}{1+\cos\theta}$.

(iii) $\sqrt{\left\{\dfrac{1-\sin\theta}{1+\sin\theta}\right\}}=\dfrac{1-\sin\theta}{\cos\theta}$.

(iv) $\sqrt{\left\{\dfrac{1+\sin^2\theta\sec^2\theta}{1+\cos^2\theta\csc^2\theta}\right\}}=\tan\theta$.

4. State two values of θ for which the statement of exercise 1(vii) is not meaningful.

5. State two values of θ for which the statement of exercise 1(x) is not meaningful.

6. State two values of θ for which the statement of exercise 2(ii) is meaningful but not true.

7. State two values of θ for which each side of the statement in exercise 2(ii) has a meaning, but for which the equality does not hold.

2.8. To Find The Values Of The Trigonometric Functions Of A Given Angle In Terms Of The Value Of One Trigonometric Function Of The Same Angle

We have already seen how we can compute the values of the trigonometric functions of an angle when the co-ordinates of a point on the terminal ray of the angle are given. Another problem of a similar nature is to find the values of all the trigonometric functions of an angle when the value of one of them is given. For example, let $\cos\theta = -3/5$, and let us be required to find the values of the remaining trigonometric functions of θ. We have to first observe that since $\cos\theta$ is negative, therefore θ must be either in the second quadrant or in the third quadrant.

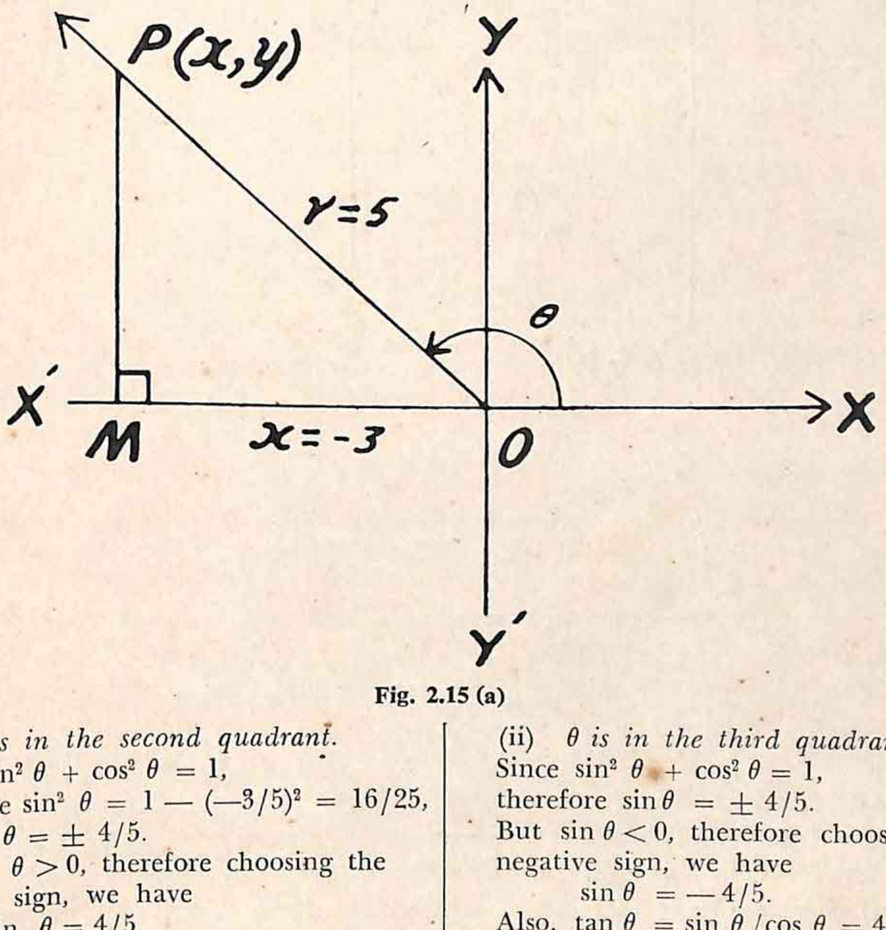

Fig. 2.15 (a)

(i) *θ is in the second quadrant.*
Since $\sin^2\theta + \cos^2\theta = 1$,
therefore $\sin^2\theta = 1 - (-3/5)^2 = 16/25$,
i.e., $\sin\theta = \pm 4/5$.
But $\sin\theta > 0$, therefore choosing the positive sign, we have

$$\sin\theta = 4/5.$$

Also,
$$\tan\theta = \sin\theta/\cos\theta, = -4/3,$$
$$\cot\theta = 1/\tan\theta = -3/4,$$
$$\sec\theta = 1/\cos\theta = -5/3,$$
$$\csc\theta = 1/\sin\theta = 5/4.$$

(ii) *θ is in the third quadrant.*
Since $\sin^2\theta + \cos^2\theta = 1$,
therefore $\sin\theta = \pm 4/5$.
But $\sin\theta < 0$, therefore choosing the negative sign, we have

$$\sin\theta = -4/5.$$

Also,
$$\tan\theta = \sin\theta/\cos\theta = 4/3,$$
$$\cot\theta = 1/\tan\theta = 3/4,$$
$$\sec\theta = 1/\cos\theta = -5/3,$$
$$\csc\theta = 1/\sin\theta = -5/4.$$

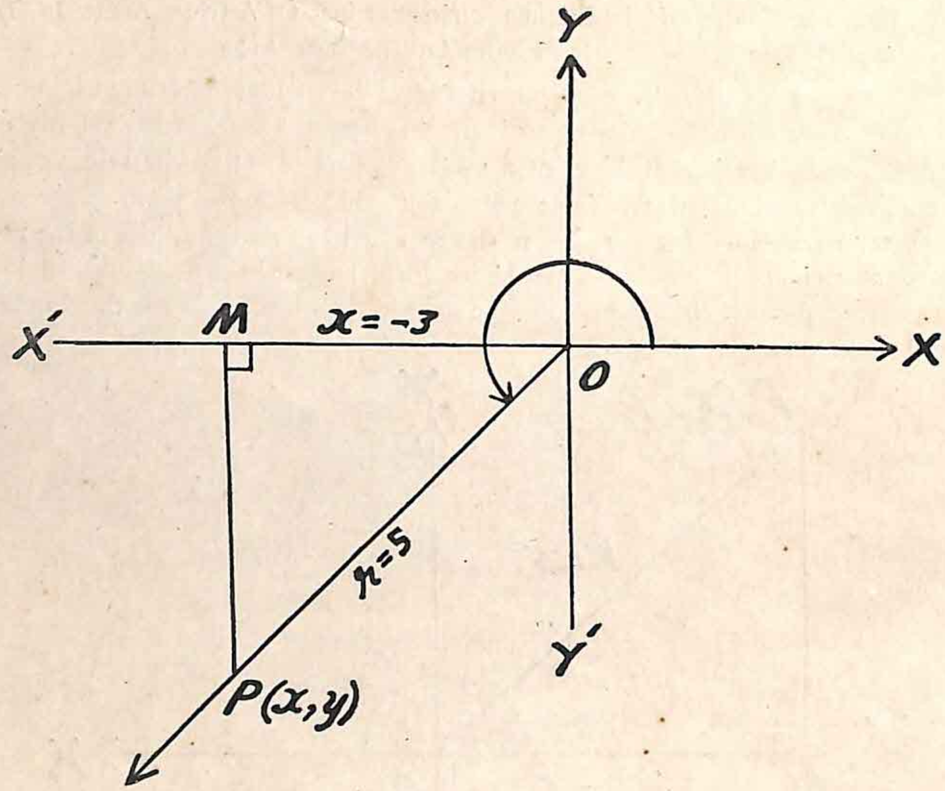

Fig. 2.15 (b)

We could also have proceeded thus :

(i) In figure 2.15(a), by observing that $x^2 + y^2 = r^2$, $x = -3$, $r = 5$, and $y > 0$, we find that $y = 4$.

Therefore $\sin\theta = y/r = 4/5$, $\tan\theta = y/x = -4/3$, $\cot\theta = x/y = -3/4$, $\sec\theta = -5/3$, $\csc\theta = r/y = 5/4$.

(ii) In figure 2.15(b) by observing $x^2 + y^2 = r^2$, $x = -3$, $r = 5$, and $y < 0$, we find that $y = -4$.

Therefore, $\sin\theta = y/r = -4/5$, $\tan\theta = y/x = 4/3$, $\cot\theta = x/y = 3/4$, $\sec\theta = r/x = -5/3$, and $\csc\theta = r/y = -5/4$.

If instead of the value of $\cos\theta$ we were given the value of some other trigonometric function, then also either of the methods could be applied (with suitable modifications of course !) to compute the values of the remaining functions. Of the two methods discussed above, the first is more suited to theoretical problems, and the second is more suited to numerical problems.

We now propose to discuss as to how we can deal with such a problem (of finding the values of all the trigonometric functions of an angle when the value of one of them is given) in general.

I *In terms of the value of the cosine.*

Since $\cos^2\theta + \sin^2\theta = 1$,

therefore $\sin\theta = \pm\sqrt{(1-\cos^2\theta)}$.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\pm\sqrt{(1-\cos^2\theta)}}{\cos\theta},$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\cos\theta}{\pm\sqrt{(1-\cos^2\theta)}},$$

$$\sec\theta = 1/\cos\theta,$$

$$\csc\theta = 1/\sin\theta = \frac{1}{\pm\sqrt{(1-\cos^2\theta)}}.$$

II *In terms of the value of the sine.*

Since $\cos^2\theta + \sin^2\theta = 1$,

therefore $\cos\theta = \pm\sqrt{(1-\sin^2\theta)}$.

Also, $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{\sin\theta}{\pm\sqrt{(1-\sin^2\theta)}}$,

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\pm\sqrt{(1-\sin^2\theta)}}{\sin\theta},$$

$$\sec\theta = 1/\cos\theta = \frac{1}{\pm\sqrt{(1-\sin^2\theta)}},$$

$$\csc\theta = 1/\sin\theta.$$

III *In terms of the value of the tangent.*

It can be easily verified (please verify !) that

$$\cos\theta = \frac{1}{\pm\sqrt{(1+\tan^2\theta)}},\ \sin\theta = \frac{\tan\theta}{\pm\sqrt{(1+\tan^2\theta)}},$$

$$\cot\theta = 1/\tan\theta,\ \sec\theta = \pm\sqrt{(1+\tan^2\theta)},$$

$$\csc\theta = \frac{\pm\sqrt{(1+\tan^2\theta)}}{\tan\theta}.$$

IV *In terms of the value of the cotangent.*

It can be easily verified (please verify !) that

$$\cos\theta = \frac{\cot\theta}{\pm\sqrt{(1+\cot^2\theta)}},\ \sin\theta = \frac{1}{\pm\sqrt{(1+\cot^2\theta)}},$$

$$\sec\theta = \frac{\pm\sqrt{(1+\cot^2\theta)}}{\cot\theta},\ \csc\theta = \pm\sqrt{(1+\cot^2\theta)}.$$

V *In terms of the value of the cosecant.*

It can be easily verified (please verify !) that

$$\cos\theta = \frac{\pm\sqrt{(\csc^2\theta - 1)}}{\csc\theta},\ \sin\theta = 1/\csc\theta,$$

$$\tan\theta = \frac{1}{\pm\sqrt{(\csc^2\theta - 1)}},$$

$$\cot\theta = \pm\sqrt{(\csc^2\theta - 1)},\ \sec\theta = \frac{\csc\theta}{\pm\sqrt{(\csc^2\theta - 1)}}.$$

VI *In terms of the value of secant.*

It can be easily verified (please verify) !) that

$$\cos\theta = 1/\sec\theta\,, \qquad \sin\theta = \pm\frac{\sqrt{(\sec^2\theta - 1)}}{\sec\theta},$$

$$\tan\theta = \pm\sqrt{(\sec^2\theta - 1)},\ \cot\theta = \pm\frac{1}{\sqrt{(\sec^2\theta - 1)}},$$

$$\csc\theta = \frac{\sec\theta}{\pm\sqrt{(\sec^2\theta - 1)}}.$$

Remark. In the various expressions given above, the signs + and – correspond to the two possible positions of the terminal ray of the angle for any given value of one of the trigonometric functions of the angle. The sign can be determined if in addition to knowing the value of one trigonometric function of the angle, we also know the quadrant in which the angle lies. The following example will illustrate the point.

Example 10. Given $\cot\theta = -3$ and θ lies in the fourth quadrant, find the values of the remaining trigonometric functions of θ.

Solution. Since $\csc^2\theta = 1 + \cot^2\theta$ and $\cot\theta = -3$, therefore $\csc^2\theta = 1 + (-3)^2 = 10$, or $\csc\theta = \pm\sqrt{10}$. Also, since θ lies in the fourth quadrant, therefore $\csc\theta$ must be negative. Therefore, we must have $\csc\theta = -\sqrt{10}$.

Again, since $\sin\theta = 1/\csc\theta$,

therefore $\sin\theta = -1/\sqrt{10}$.

Also, $\cos\theta = \cot\theta\,.\,\sin\theta = (-3)(-1/\sqrt{10}) = 3/\sqrt{10}$,

$\tan\theta = 1/\cot\theta = -1/3$,

$\sec\theta = 1/\cos\theta/ = \sqrt{10}/3$.

Example. Given $\tan\theta = 0$, find the values of the other trigonometric functions of θ.

Solution. If (x, y) be the co-ordinates of any point P on the terminal ray of the angle θ, then $y/x = 0$. Since $\tan\theta = 0$, therefore $y = 0$. Hence P must lie on the x-axis. The terminal ray should, therefore, lie along the positive x-axis or the negative x-axis. The angle θ is either 0° or 180°.

If $\theta = 0°$, then

$\sin\theta = 0$,

$\cos\theta = 1$,

$\sec\theta = 1$,

$\cot\theta$ and $\csc\theta$ are not defined.

If $\theta = 180°$, then

$\sin\theta = 0$,

$\cos\theta = -1$,

$\sec\theta = -1$.

$\cot\theta$ and $\csc\theta$ are not defined.

EXERCISES

1. Construct two angles in standard position which satisfy the following condition, and write down the values of the other trigonometric functions.

(i) $\sin\theta = 5/13$.
(ii) $\cos\theta = -4/5$.
(iii) $\tan\theta = 24/7$.
(iv) $\cot\theta = 0$.
(v) $\sec\theta = 3$.
(vi) $\csc\theta = -4$.

2. Construct a positive angle less than 360° which satisfies the conditions in each of the following problems, and write down the values of the other trigonometric functions :

(i) $\cos\theta = 4/5$, θ lies in the first quadrant.
(ii) $\sin\theta = 5/13$, θ lies in the second quadrant.
(iii) $\tan\theta = -15/8$, θ lies in the fourth quadrant.
(iv) $\cot\theta = -4/3$, θ lies in the second quadrant.
(v) $\csc\theta = -13/12$, θ lies in the third quadrant.
(vi) $\sec\theta = 25/7$, θ lies in the fourth quadrant.
(vii) $\cos\theta = 1/2$, θ does not lie in the first quadrant.
(viii) $\sin\theta = 3/4$, θ does not lie in the second quadrant.
(ix) $\tan\theta = 3/2$, θ does not lie in the third quadrant.
(x) $\cot\theta = -5/6$, θ does not lie in the fourth quadrant.
(xi) $\csc\theta = 3$, θ does not lie in the first quadrant.
(xii) $\sec\theta = -5/2$, θ does not lie in the third quadrant.
(xiii) $\cos\theta = -5/13$, $\sin\theta > 0$.
(xiv) $\sin\theta = -1/2$, $\tan\theta > 0$.
(xv) $\tan\theta = 2$, $\cos\theta < 0$.
(xvi) $\cot\theta = -3/4$, $\sec\theta > 0$.
(xvii) $\csc\theta = -25/7$, $\cot\theta > 0$.
(xviii) $\sec\theta = 13/12$, $\csc\theta < 0$.

3. If $\sin\theta = 12/13$ and θ lies in the second quadrant, find the value of $\sec\theta + \tan\theta$.

4. In which quadrants may θ lie, in order that the following statements may be true :

(i) $\sin\theta = \sqrt{(1 - \cos^2\theta)}$.
(ii) $\cos\theta = -\sqrt{(1 - \sin^2\theta)}$.
(iii) $\tan\theta = \sqrt{(\sec^2\theta - 1)}$.
(iv) $\sec\theta = -\sqrt{(1 + \tan^2\theta)}$.
(v) $\csc\theta = \sqrt{(\cot^2\theta + 1)}$.
(vi) $\cot\theta = -\sqrt{(\csc^2\theta - 1)}$.

5. Prove or disprove the following statements :

(i) $\cot\theta = 3/4 \wedge \sin\theta < 0 \Rightarrow \cos\theta = -3/5$.
(ii) $\sin\theta = -5/7 \wedge \sec\theta > 0 \Rightarrow \tan\theta = 1$.

2.9. Even and Odd Functions

Consider the function f given by $f(x) = x^2$. Since $(-x)^2 = x^2$, for all x, therefore $f(-x) = (-x)^2 = x^2 = f(x)$, for all x. Similarly for each of the functions g and h defined by $g(x) = |x|$, $h(x) = x^4$, we have $g(-x) = g(x)$ and $h(-x) = h(x)$, for all x. Such functions are called *even functions*.

Definition. A function f is said to be even if

$$f(-x) = f(x),$$

for all x in the domain of f.

Next let us consider the function F given by $F(x) = x^3$. Since $(-x)^3 = -x^3$ for all x, therefore $F(-x) = (-x)^3 = -x^3 = -F(x)$, for all x. Similarly for the functions G and H given by $G(x) = x^5$ and $H(x) = x^7$, we have $G(-x) = -G(x)$, $H(-x) = -H(x)$ for all x. Such functions are called *odd functions.*

Definition. A function f is said to be odd if

$$f : f(x) = x^2 + x.$$

for all x in the domain of f.

Finally, let us consider the function

$$f : f(x) = x^2 + x.$$

Here $f(-x) = x^2 - x$ which is neither equal to $f(x)$ nor to $-f(x)$ for any x other than $x = 0$. This function is therefore neither ever nor odd. Similarly the functions

$$g : g(x) = x^2 - 3x + 2,$$
$$h : h(x) = x^3 - 2x + 1,$$

are neither odd nor even.

We shall now examine as to which of the trigonometric functions are odd, even, or neither odd nor even.

Theorem (i). The cosine and the secant are even functions, i.e.,

$$\cos(-\theta) = \cos\theta,$$
$$\sec(-\theta) = \sec\theta.$$

(ii). The sine, the tangent, the cotangent and the cosecant are odd functions, i.e.,

$$\sin(-\theta) = -\sin\theta, \ \csc(-\theta) = -\csc\theta,$$
$$\tan(-\theta) = -\tan\theta, \ \cot(-\theta) = -\cot\theta.$$

Proof. Let θ be *any* given angle. We consider the angle $-\theta$. Let P and Q be the points on the unit circle where the terminal sides of these angles intersect it. Since rotations of angles θ, $-\theta$ take the initial line OA to OP and OQ respectively, therefore OP, OQ must be symmetrical with respect to OA, i.e., the x-axis. Consequently if the co-ordinates of P are (x, y), the co-ordinates of Q are (x, y).

Thus we have

$$\cos\theta = x, \sin\theta = y,$$
$$\text{and } \cos(-\theta) = x, \sin(-\theta) = -y.$$

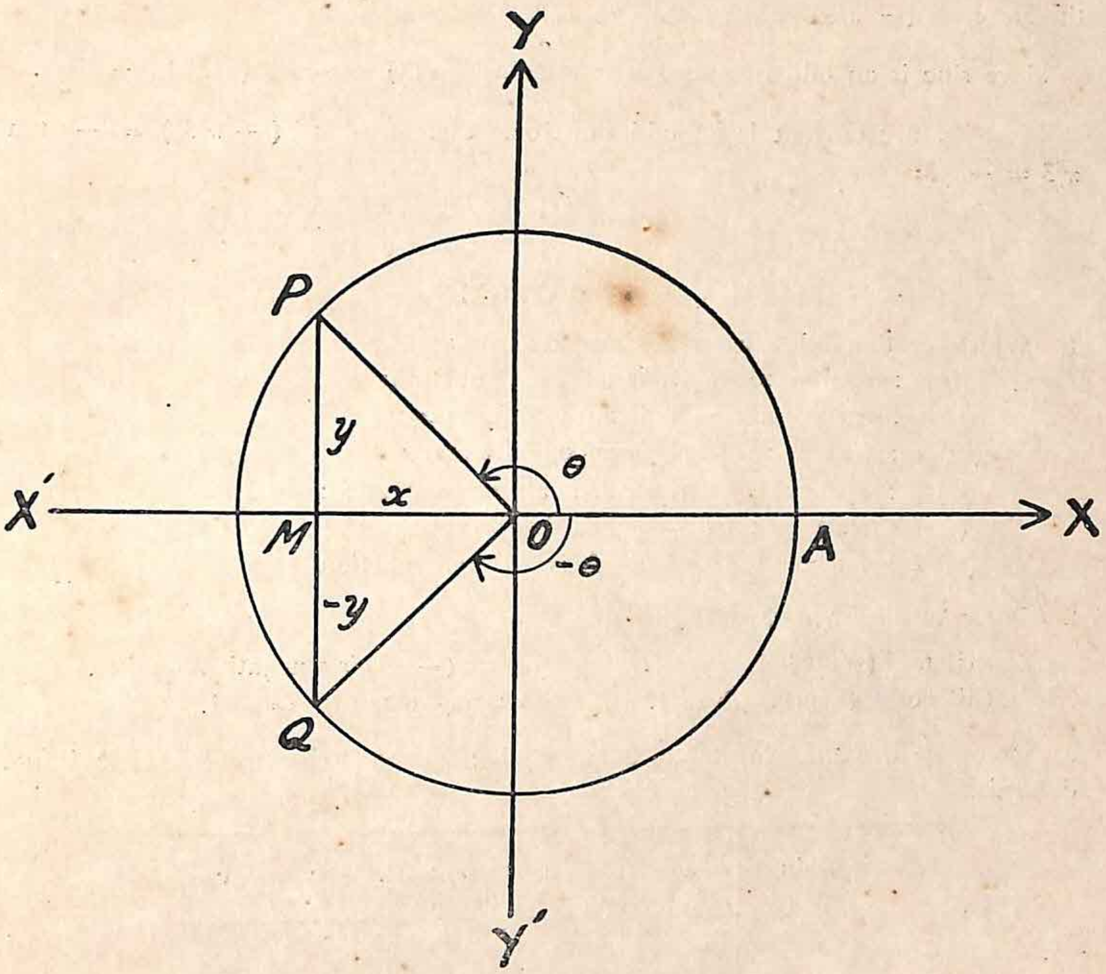

Fig. 2.16

Consequently,

$\cos(-\theta) = \cos\theta$, $\sin(-\theta) = -\sin\theta$.

From these we get

$\sec(-\theta) = 1/\cos(-\theta) = 1/\cos\theta = \sec\theta$,

$\tan(-\theta) = \sin(-\theta)/\cos(-\theta) = -\sin\theta/\cos\theta = -\tan\theta$,

$\cot(-\theta) = \cos(-\theta)/\sin(-\theta) = \cos\theta/-\sin\theta = -\cot\theta$,

$\csc(-\theta) = 1/\sin(-\theta) = 1/(-\sin\theta) = -\csc\theta$.

Example. Compute $\cos(-\pi/3)$, $\sin(-\pi/3)$ and $\tan(-\pi/3)$.

Solution. Since cosine is an even function, therefore $\cos(-\pi/3) = \cos(\pi/3) = 1/2$.

Since sine is an odd function, therefore $\sin(-\pi/3) = -\sin(\pi/3) = -\frac{\sqrt{3}}{2}$.

Since tangent is an odd function, therefore $\tan(-\pi/3) = -\tan\pi/3 = -\sqrt{3}$.

EXERCISES

1. Which of the following statements are true :
   - (i) $f : f(x) = x^2 + 1$, is an even function;
   - (ii) $f : f(x) = x^3 - 1$, is an odd function;
   - (iii) $f : f(x) = x^5 + x$, is an odd function;
   - (iv) $f : f(\theta) = \cos\theta + \sec\theta$, is an even function;
   - (v) $f : f(\theta) = \tan\theta - \sin\theta$, is an odd function;
   - (vi) $f : f(\theta) = \tan\theta + \sec\theta$, is an even function.

2. Prove or disprove the following :
   - (i) $\tan^2(-\theta) + 1 = \sec^2\theta$.
   - (ii) $\cos(-\pi/3)$ is a negative number.
   - (iii) $\cos^2\theta + \sin^2(-\theta) = 1$.
   - (iv) $\tan(-\theta)\cot(\theta) = -1$.

3. Complete the following table. Put a × whenever the trigonometric ratio is not defined.

	sin	cos	tan	cot	sec	csc
$-90°$						
$-60°$						
$-45°$						
$-30°$						

4. Find the value of each of the following expressions:
   - (i) $2\sin(-30°) + \cos^2(-45°) + \tan(-30°)\sin 30°$,
   - (ii) $\cos(-\pi/4)\cos(-\pi/2) + \sin(-\pi/3)\sin\pi/3$.
   - (iii) $\cos^2(-\pi/3) - \sin^2(-\pi/6) + 1 - \sin^2(-\pi/2)$,

ASSORTED EXERCISES ON CHAPTER 2

1. The terminal ray of an angle θ in standard position passes through the point $(24, -7)$. Draw the reference triangle, make the angle θ, and write down all the trigonometric ratios of θ.

2. Prove that $(a + b)^2 = 4ab \sin^2\theta$ is a possible equation only when $a = b$.

3. Show that no value of $\sec\theta$ can satisfy the equation $6 \sec^2\theta - 5 \sec\theta + 1 = 0$.

4. Show that the equation $\sec^2\theta = 4xy/(x + y)^2$ is a possible equation only when $x = y$.

5. Prove that $\sec^2\theta + \cos^2\theta$ can never be less than 2.

6. In which quadrant (or quadrants) must θ be in order to satisfy the following statements :

    (i) $\sin\theta < 0 \wedge \tan\theta < 0$.
    (ii) $\cos\theta > 0 \wedge \cot\theta < 0$.
    (iii) $\tan\theta < 0$.
    (iv) $\sin\theta > 0$.

7. Name two trigonometric ratios which are not defined when $\theta = 180°$.

8. Construct an angle of 330° in standard position and find the values of all its trigonometric ratios.

9. Prove the following identities :

    (i) $(\sin\theta + \sec\theta)^2 + (\cos\theta + \csc\theta)^2 = (1 + \sec\theta \operatorname{cse}\theta)^2$.
    (ii) $\csc^6\theta - \cot^6\theta = 1 + 3 \csc^2\theta \cot^2\theta$.
    (iii) $\cos\theta/(1 - \tan\theta) + \sin\theta/(1 - \cot\theta) = \sin\theta + \cos\theta$.

10. If $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$, prove that $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$.

11. Prove or disprove the following statement :

$$\cos\theta = -4/5 \wedge \sin\theta > 0 \Rightarrow \tan\theta = -3/4.$$

12. If $\sin\theta = 12/13$ and θ lies in the second quadrant, find the value of $\sec\theta + \tan\theta$.

13. If θ is a positive acute angle, the terminal ray of an angle of $360° - \theta$ in standard position is in the fourth quadrant. Use the adjoining figure to show that in this case

$$\begin{aligned}
\sin(360° - \theta) &= -\sin\theta,\\
\cos(360° - \theta) &= \cos\theta,\\
\tan(360° - \theta) &= -\tan\theta,\\
\cot(360° - \theta) &= -\cot\theta,\\
\sec(360° - \theta) &= \sec\theta,\\
\csc(360° - \theta) &= -\csc\theta.
\end{aligned}$$

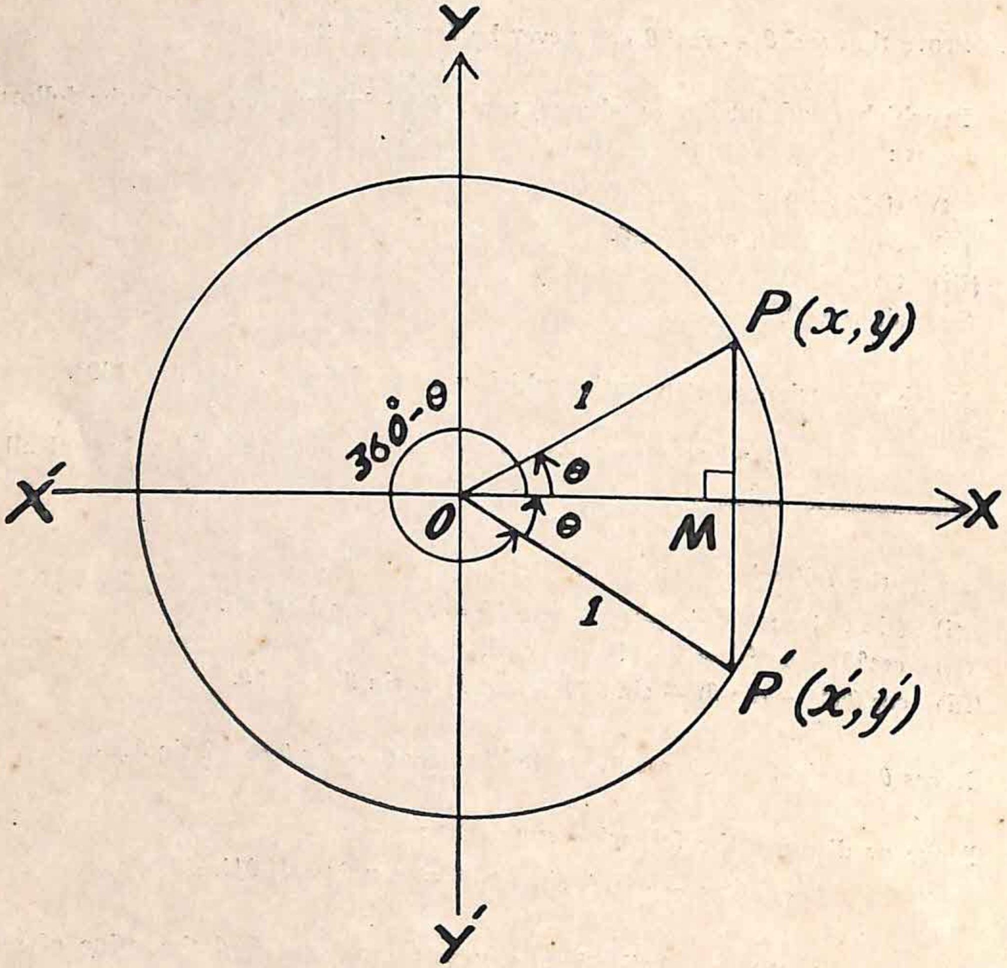

Fig. 2.17

14. If θ is a positive acute angle, the terminal ray of an angle of $180° + \theta$ in standard position is in the third quadrant. Use figure 2.18 to show that in this case

$$\begin{aligned}
\sin(180° + \theta) &= -\sin\theta, \\
\cos(180° + \theta) &= -\cos\theta, \\
\tan(180° + \theta) &= \tan\theta, \\
\cot(180° + \theta) &= \cot\theta, \\
\sec(180° + \theta) &= -\sec\theta, \\
\csc(180° + \theta) &= -\csc\theta.
\end{aligned}$$

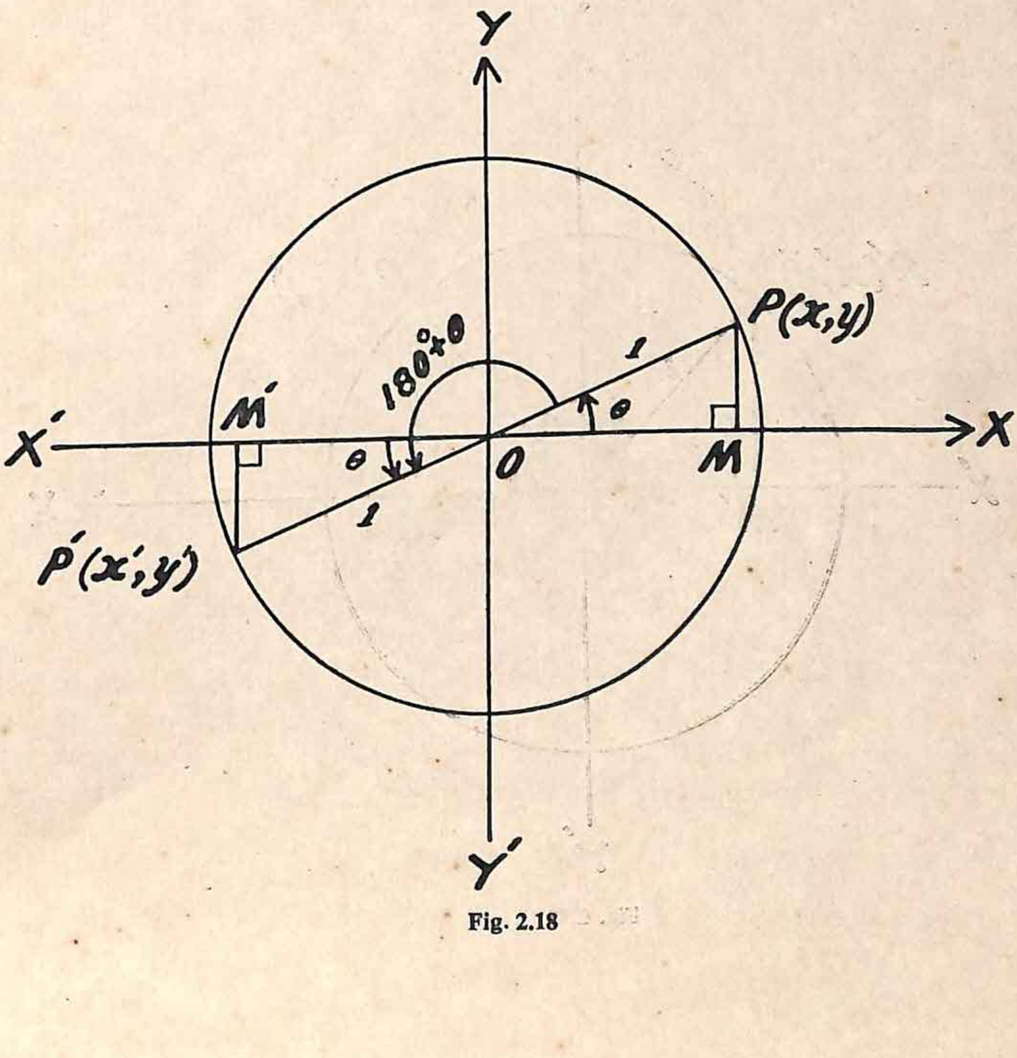

Fig. 2.18

15. If θ is a positive acute angle, the terminal ray of an angle of $180° - \theta$ in standard position is in the second quadrant. Use the figure 2.19 to show that in this case

$$\begin{aligned}
\sin(180° - \theta) &= \sin\theta, \\
\cos(180° - \theta) &= -\cos\theta, \\
\tan(180° - \theta) &= -\tan\theta, \\
\cot(180° - \theta) &= -\cot\theta, \\
\sec(180° - \theta) &= -\sec\theta, \\
\csc(180° - \theta) &= \csc\theta,
\end{aligned}$$

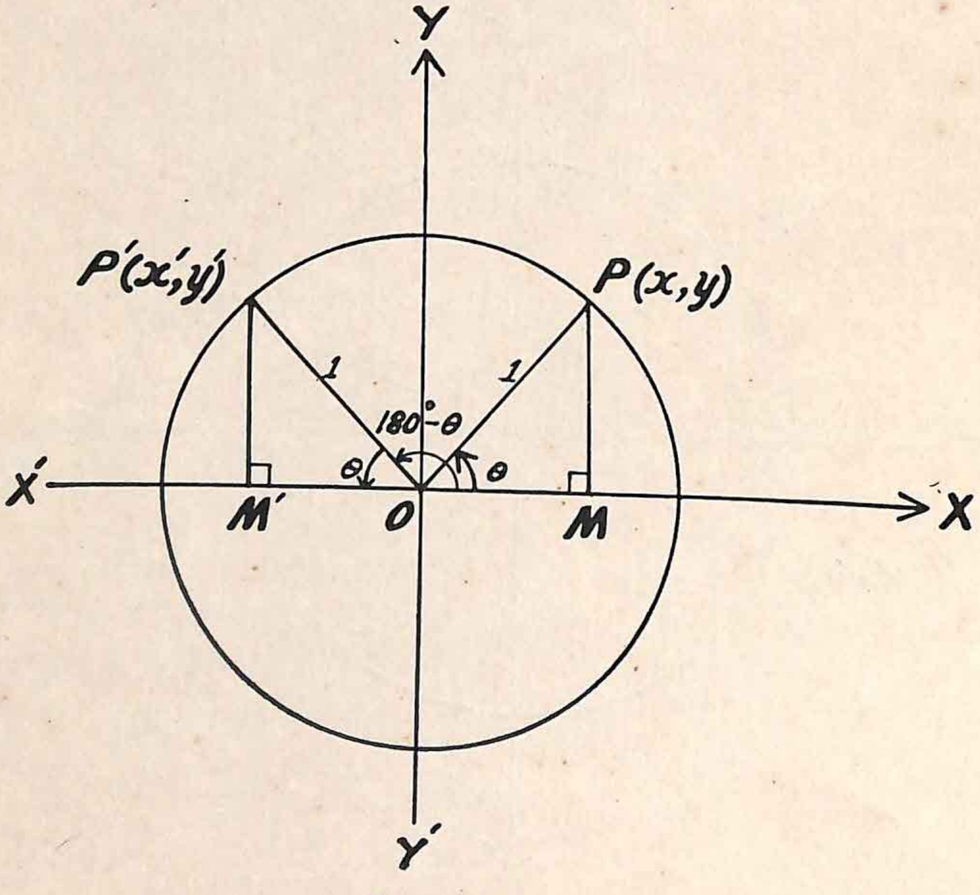

Fig. 2.19

16. If θ is a positive acute angle, the terminal ray of an angle of $90° + \theta$ in standard position is in the second quadrant. Use the figure 2.20 to show that in this case

$$\begin{aligned}
\sin(90° + \theta) &= \cos\theta,\\
\cos(90° + \theta) &= -\sin\theta,\\
\tan(90° + \theta) &= -\cot\theta,\\
\cot(90° + \theta) &= -\tan\theta,\\
\sec(90° + \theta) &= -\csc\theta,\\
\csc(90° + \theta) &= \sec\theta.
\end{aligned}$$

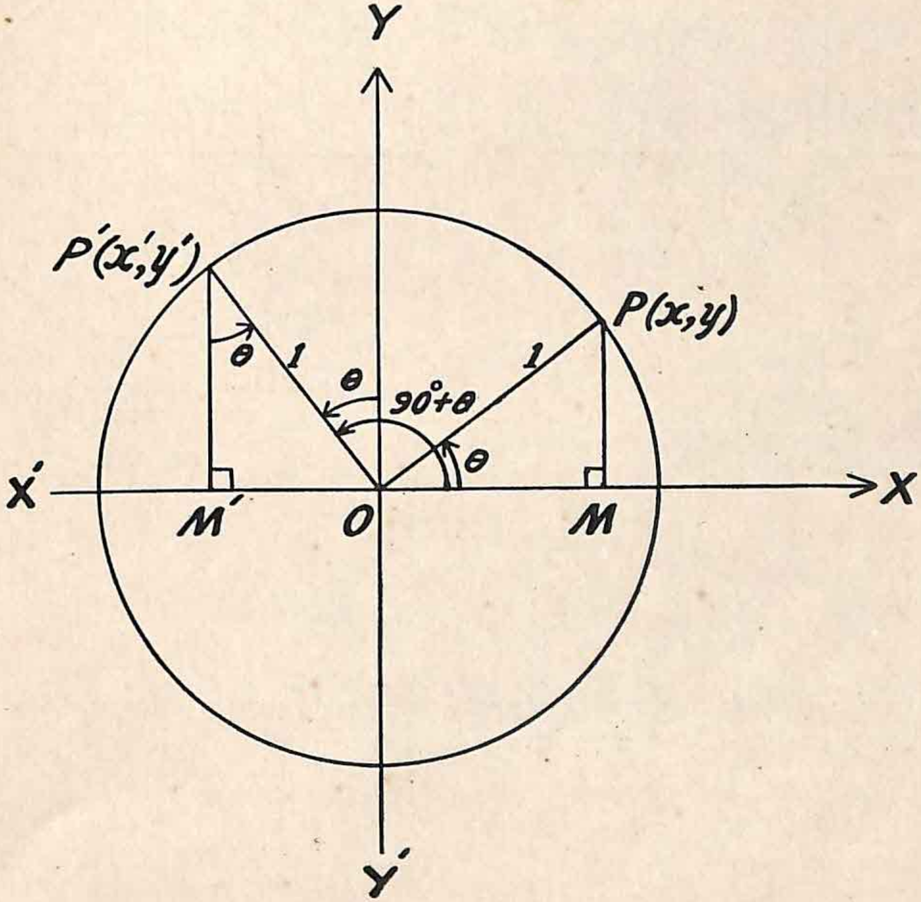

Fig. 2.20

17. Use the following figures to express the trigonometric functions of $(270° + \theta)$ and $(270° - \theta)$ as functions of θ, θ being a positive acute angle.

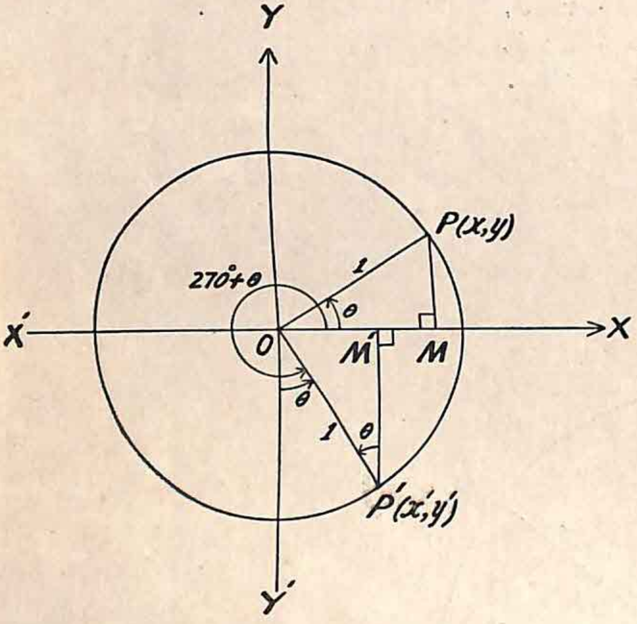

Fig. 2.21

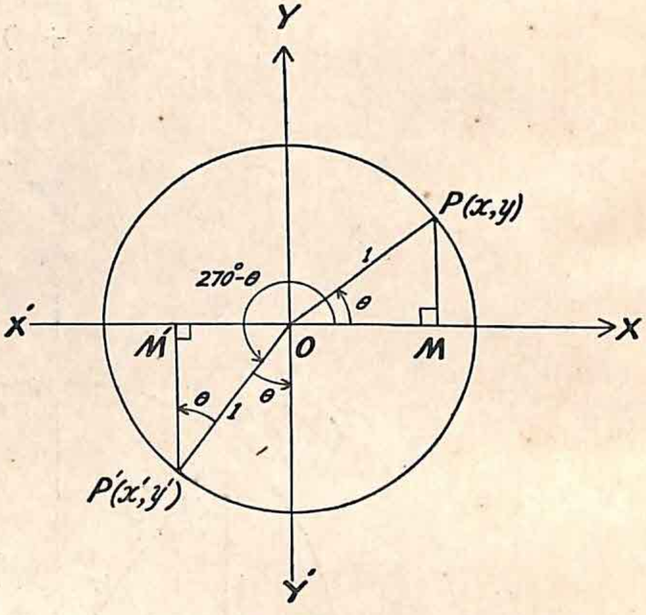

Fig. 2.22

18. Draw a suitable figure and express the trigonometric ratios of $(90° - \theta)$ as functions of θ, θ being a positive acute angle.

Chapter 3

Addition Theorems for Trigonometric Functions

3.1. Introduction

In the present chapter we shall obtain expressions for trigonometric functions of the sum of two angles in terms of the trigonometric functions of the angles and discuss consequences of the same.

3.2. Addition Theorem for the Cosine

Theorem. For any two angles θ and ϕ,

$$\cos(\theta + \phi) = \cos\theta \cos\phi - \sin\theta \sin\phi .$$

Proof. Let us draw θ in standard position and ϕ with its initial side along the terminal side of θ, as shown in figure 3.1. Thus $\theta + \phi$ is in standard position. Also, $-\phi$ is in standard position, and the angle $-(-\phi) + \theta = \phi + \theta$ is as shown. Let us draw a unit circle with the origin O as centre. Let P_1, P_2, P_3 be the points of intersection of the terminal sides of the angles $\theta, \theta + \phi, -\phi$ with the unit circle. Also let P_4 be the point of intersection of the positive x-axis with the unit circle. Then P_1, P_2, P_3 have co-orordinates $(\cos\theta, \sin\theta)$ $\{\cos(\theta + \phi), \sin(\theta + \phi)\}$ $\{\cos(-\phi), \sin(-\phi)\}$ respectively. Also P_4 has co-ordinates $(1,0)$.

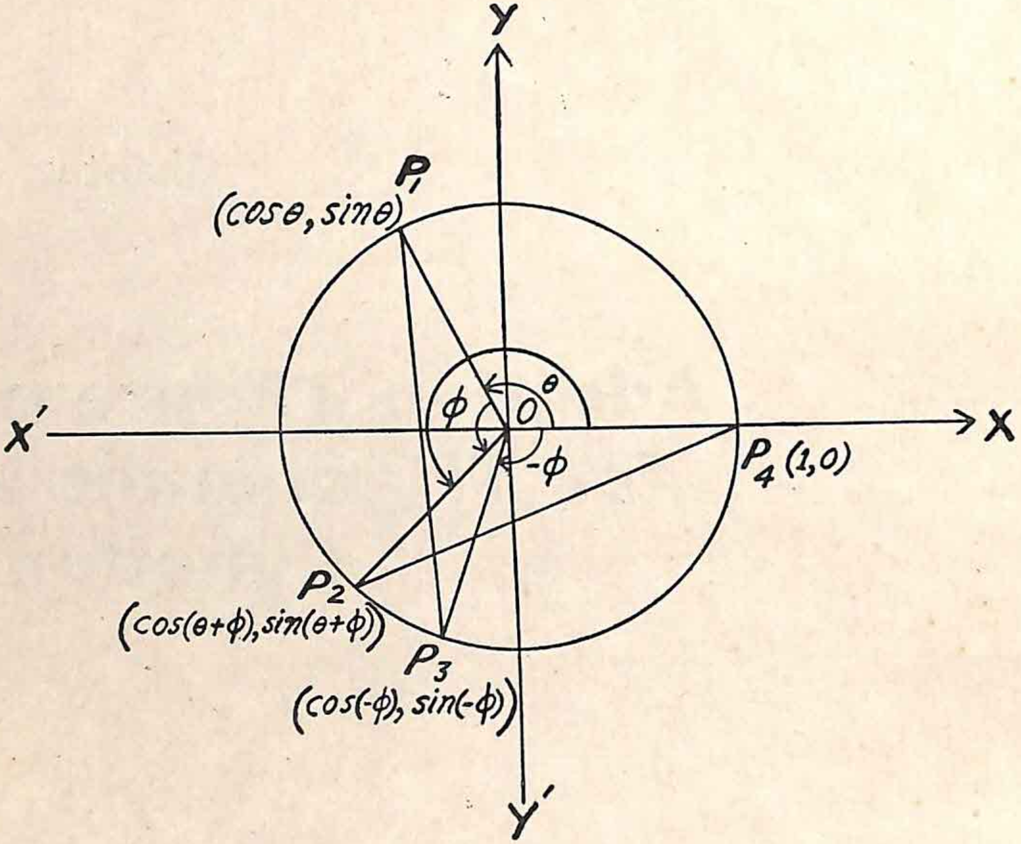
Y
P_1
$(\cos\theta, \sin\theta)$
ϕ
θ
O
$-\phi$
X′
X
$P_4\,(1,0)$
P_2
$(\cos(\theta+\phi), \sin(\theta+\phi))$
P_3
$(\cos(-\phi), \sin(-\phi))$
Y′

Fig. 3.1

The triangles P_1OP_3 and P_2OP_4 are congruent since two sides and included angle of the one are equal to the two sides and included angle of the other. Therefore P_2P_4 and P_1P_3 are equal. By the distance formula,

$$P_1P_3^2 = \{\cos\theta - \cos(-\phi)\}^2 + \{\sin\theta - \sin(-\phi)\}^2$$

$= (\cos\theta - \cos\phi)^2 + (\sin\theta + \sin\phi)^2$, since $\cos(-\theta) = \cos\theta$, $\sin(-\phi) = -\sin\phi$,

$= (\cos^2\theta - 2\cos\theta\cos\phi + \cos^2\phi) + (\sin^2\theta + 2\sin\theta\sin\phi + \sin^2\phi)$,

$= 2 - (2\cos\theta\cos\phi - \sin\theta\sin\phi)$, since $\cos^2\theta + \sin^2\theta = 1$, $\cos^2\phi + \sin^2\phi = 1$.

Also $P_2P_4^2 = \{1 - \cos(\theta + \phi)\}^2 + \{0 - \sin(\theta + \phi)\}^2$

$= 1 - 2\cos(\theta + \phi) + \cos^2(\theta + \phi) + \sin^2(\theta + \phi)$

$= 2 - 2\cos(\theta + \phi)$, since $\cos^2(\theta + \phi) + \sin^2(\theta + \phi) = 1$.

Equating the value of $P_2P_4^2$ and $P_1P_3^2$ we have,

$$2 - 2\cos(\theta + \phi) = 2 - 2(\cos\theta\cos\phi - \sin\theta\sin\phi). \qquad \ldots\ldots \text{(i)}$$

Subtracting 2 from both sides of (i) and dividing each side of the resulting equation by —2, we have

$$\boxed{\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi.}$$

The above proof is obviously valid for all values of θ and ϕ.

Corollary. $\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi.$

Proof.

$$\begin{aligned}\cos(\theta - \phi) &= \cos[\theta + (-\phi)],\\ &= \cos\theta\cos(-\phi) - \sin\theta\sin(-\phi),\\ &= \cos\theta\cos\phi - \sin\theta(-\sin\phi),\\ [\text{since } \cos(-\theta) &= \cos\theta, \sin(-\phi) = -\sin\phi]\\ &= \cos\theta\cos\phi + \sin\theta\sin\phi.\end{aligned}$$

3.3. Formulae for Complementary Angles

Two angles θ and ϕ are said to be complementary if $\theta + \phi = \pi/2$. This is a generalisation of the usual notion of complementary angles for positive acute angles.

Theorem. The cosine of any angle is equal to the sine of its complement. In symbols,

if $\theta + \phi = \pi/2$, then $\cos\phi = \sin\theta$, $\cos\theta = \sin\phi$,

or $$\boxed{\cos(\pi/2 - \theta) = \sin\theta,\ \sin(\pi/2 - \theta) = \cos\theta}$$

Proof.

$$\begin{aligned}\cos\phi &= \cos(\pi/2 - \theta),\\ &= \cos\pi/2\cos\theta + \sin\pi/2\sin\theta,\\ &= 0.\cos\theta + 1.\sin\theta,\\ &= \sin\theta.\end{aligned}$$

Similarly $\cos\theta = \sin\phi$.

Corollary. $$\boxed{\begin{aligned}&\tan(\pi/2 - \theta) = \cot\theta, \quad \cot(\pi/2 - \theta) = \tan\theta,\\ &\sec(\pi/2 - \theta) = \operatorname{cosec}\theta,\ \operatorname{cosec}(\pi/2 - \theta) = \sec\theta,\end{aligned}}$$

whenever both sides have a meaning.

3.4. Addition Theorem for the Sine

Theorem. For any two angles θ and ϕ,

$$\boxed{\sin(\theta + \phi = \sin\theta.\cos\phi + \cos\theta.\sin\phi.} \quad \text{......(i)}$$

Proof.

$$\begin{aligned}\sin(\theta + \phi) &= \cos\{\pi/2 - (\theta + \phi)\}, \text{ by Art. 3.3}\\ &= \cos\{(\pi/2 - \theta) - \phi\},\end{aligned}$$

$= \cos(\pi/2 - \theta)\cos\phi + \sin(\pi/2 - \theta)\sin\phi$, by Art. 3·2

$= \sin\theta\cos\phi + \cos\theta\sin\phi$, since $\cos(\pi/2 - \theta) = \sin\theta$ and $\sin(\pi/2 - \theta) = \cos\theta$.

Corollary. For any two angles θ and ϕ,

$$\boxed{\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi.} \qquad \ldots\ldots(2)$$

Proof. $\sin(\theta - \phi) = \sin[\theta + (-\phi)]$,

$= \sin\theta\cos(-\phi) + \cos\theta\sin(-\phi)$, by (1) above

$= \sin\theta\cos\phi - \cos\theta\sin\phi$, by theorem 3.3.

Example 1. Calculate sin 15° and sin 75°.

Solution.
$$\begin{aligned}\sin 15^\circ &= \sin(45^\circ - 30^\circ)\\ &= \sin 45^\circ\cos 30^\circ - \cos 45^\circ\sin 30^\circ,\\ &= \frac{1}{\sqrt{2}}.\,\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}.\,\frac{1}{2},\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}},\\ \sin 75^\circ &= \sin(45^\circ + 30^\circ),\\ &= \sin 45^\circ\cos 30^\circ + \cos 45^\circ\sin 30^\circ,\\ &= \frac{1}{\sqrt{2}}.\,\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}.\,\frac{1}{2},\\ &= \frac{\sqrt{3}+1}{2\sqrt{2}}.\end{aligned}$$

3.5. Addition Theorem for the Tangent

Theorem. If none of the angles θ, ϕ and $\theta + \phi$ is an odd multiple of $\frac{\pi}{2}$, then,

$$\boxed{\tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}.}$$

Proof. As none of the angles $\theta, \phi, \theta + \phi$ is an odd multiple of $\frac{\pi}{2}$, therefore $\cos\theta\cos\phi \neq 0$ and $\cos(\theta + \phi) \neq 0$.

$$\begin{aligned}\tan(\theta + \phi) &= \frac{\sin(\theta + \phi)}{\cos(\theta + \phi)},\\ &= \frac{\sin\theta\cos\phi + \cos\theta\sin\phi}{\cos\theta\cos\phi - \sin\theta\sin\phi},\\ &= \frac{\dfrac{\sin\theta\cos\phi}{\cos\theta\cos\phi} + \dfrac{\cos\theta\sin\phi}{\cos\theta\cos\phi}}{\dfrac{\cos\theta\cos\phi}{\cos\theta\cos\phi} - \dfrac{\sin\theta\sin\phi}{\cos\theta\cos\phi}},\end{aligned}$$

(dividing the numerator and the denominator by $\cos\theta\cos\phi$)

$$= \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}.$$

Corollary.

$$\boxed{\tan(\theta - \phi) = \frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi},} \qquad \ldots\ldots(2)$$

where none of the angles θ, ϕ and $\theta - \phi$ is an odd multiple of $\pi/2$.

Proof.

$$\begin{aligned}
\tan(\theta - \phi) &= \tan[\theta + (-\phi)], \\
&= \frac{\tan\theta + \tan(-\phi)}{1 - \tan\theta\tan(-\phi)}, \\
&= \frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi}.
\end{aligned}$$

Example 2. Express $\tan(\pi/4 + \theta)$ and $\tan(\pi/4 - \theta)$ in terms of $\tan\theta$, stating the values of θ for which the expressions are valid.

Solution.

$$\begin{aligned}
\tan(\pi/4 + \theta) &= \frac{\tan\pi/4 + \tan\theta}{1 - \tan\pi/4\tan\theta}, \\
&= \frac{1 + \tan\theta}{1 - \tan\theta}.
\end{aligned}$$

The above expression is valid for all those values of θ for which none of the angles θ and $\pi/4 + \theta$ is an odd multiple of $\pi/2$.

$$\begin{aligned}
\tan(\pi/4 - \theta) &= \frac{\tan\pi/4 - \tan\theta}{1 + \tan\pi/4\tan\theta}, \\
&= \frac{1 - \tan\theta}{1 + \tan\theta}.
\end{aligned}$$

The above expression is valid for all those values of θ for which none of the angles θ and $\pi/4 - \theta$ is an odd multiple of $\pi/2$.

3.6. Addition Theorem for the Cotangent

Theorem. If none of the angles θ, ϕ and $\theta + \phi$ is a multiple of π, then

$$\boxed{\cot(\theta + \phi) = \frac{\cot\theta\cot\phi - 1}{\cot\theta + \cot\phi}.}$$

Proof. Since none of the angles θ, ϕ, $\theta + \phi$ is an even multiple of $\pi/2$, therefore $\sin\theta\ \sin\phi \neq 0$ and $\sin(\theta + \phi) \neq 0$.

$$\begin{aligned}
\cot(\theta + \phi) &= \frac{\cos(\theta + \phi)}{\sin(\theta + \phi)}, \\
&= \frac{\cos\theta\cos\phi - \sin\theta\sin\phi}{\sin\theta\cos\phi + \cos\theta\sin\phi},
\end{aligned}$$

$$= \frac{\dfrac{\cos\theta\cos\phi}{\sin\theta\sin\phi} - \dfrac{\sin\theta\sin\phi}{\sin\theta\sin\phi}}{\dfrac{\sin\theta\cos\phi}{\sin\theta\sin\phi} + \dfrac{\cos\theta\sin\phi}{\sin\theta\sin\phi}},$$

(dividing the numerator and the denominator by $\sin\theta\sin\phi$)

$$= \frac{\cot\theta\cot\phi - 1}{\cot\phi + \cot\theta},$$

$$= \frac{\cot\theta\cot\phi - 1}{\cot\theta + \cot\phi}.$$

Corollary.

$$\boxed{\cot(\theta - \phi) = \frac{\cot\theta\cot\phi + 1}{\cot\phi - \cot\theta},}$$

where none of the angles θ, ϕ and $\theta - \phi$ is a multiple of π.

Proof.

$$\begin{aligned}\cot(\theta - \phi) &= \cot[\theta + (-\phi)],\\ &= \frac{\cot\theta\cot(-\phi) - 1}{\cot\theta + \cot(-\phi)},\\ &= \frac{-\cot\theta\cot\phi - 1}{\cot\theta - \cot\phi},\\ &= \frac{\cot\theta\cot\phi + 1}{\cot\phi - \cot\theta}.\end{aligned}$$

Let us note that if any one of the angles θ, ϕ, $\theta - \phi$ were a multiple of π, then the addition theorem for the cotangent would not have been valid.

3.7. Addition Theorem for Several Angles

We can obtain addition theorems for several angles by repeated application of the theorems for two angles. We illustrate this by considering the angle $\theta + \phi + \psi$.

$$\begin{aligned}\sin(\theta+\phi+\psi) &= \sin[(\theta+\phi)+\psi],\\ &= \sin(\theta+\phi)\cos\psi + \cos(\theta+\phi)\sin\psi,\\ &= (\sin\theta\cos\phi + \cos\theta\sin\phi)\cos\psi,\\ &\quad + (\cos\theta\cos\phi - \sin\theta\sin\phi)\sin\psi,\\ &= \sin\theta\cos\phi\cos\psi + \cos\theta\sin\phi\cos\psi\\ &\quad + \cos\theta\cos\phi\sin\psi - \sin\theta\sin\phi\sin\psi. \qquad \text{......(1)}\end{aligned}$$

Similarly

$$\begin{aligned}\cos(\theta+\phi+\psi) &= \cos[(\theta+\phi)+\psi],\\ &= \cos(\theta+\phi)\cos\psi - \sin(\theta+\phi)\sin\psi,\\ &= (\cos\theta\cos\phi - \sin\theta\sin\phi)\cos\psi\\ &\quad - (\sin\theta\cos\phi + \cos\theta\sin\phi)\sin\psi,\\ &= \cos\theta\cos\phi\cos\psi - \sin\theta\sin\phi\cos\psi\\ &\quad - \sin\theta\cos\phi\sin\psi + \cos\theta\sin\phi\sin\psi. \qquad \text{......(2)}\end{aligned}$$

From the above two results, dividing the numerator and denominator by $\cos\theta \cos\phi \cos\psi$, we have

$$\tan(\theta+\phi+\psi) = \frac{\sin(\theta+\phi+\psi)}{\cos(\theta+\phi+\psi)},$$

$$= \frac{\tan\theta + \tan\phi + \tan\psi - \tan\theta\tan\phi\tan\psi}{1 - \tan\theta\tan\phi - \tan\phi\tan\psi - \tan\psi\tan\theta}.$$

Note that the two divisions will be permissible if and only if $\cos(\theta+\phi+\psi) \neq 0$ and $\cos\theta\cos\phi\cos\psi \neq 0$ i.e., none of the angles θ, ϕ, ψ and $\theta+\phi+\psi$ is an odd multiple of $\pi/2$.

EXERCISES

1. Show that $\cos(30° + 45°) \neq \cos 30° + \cos 45°$.
2. Show that $\sin(60° - 45°) \neq \sin 60° - \sin 45°$.
3. Verify that $\cos 70° \cos 10° + \sin 70° \sin 10° = 1/2$.
4. Verify that $\sin 64° \cos 26° + \cos 64° \sin 26° = 1$.
5. Prove the following identities :

(i) $\cos(60° + \theta) = 1/2\,(\cos\theta - \sqrt{3}\sin\theta)$

(ii) $\sin(30° + \theta) = 1/2\,(\cos\theta + \sqrt{3}\sin\theta)$.

(iii) $\sin(\pi/4 + \theta) = 1/\sqrt{2}\,(\cos\theta + \sin\theta)$.

(iv) $\cos\theta - \sin\theta = \sqrt{2}\cos(\theta + \pi/4)$.

(v) $\dfrac{\cos(\theta+\phi)}{\cos\theta\cos\phi} = 1 - \tan\theta\tan\phi$.

(vi) $\dfrac{\sin(\theta+\phi)}{\cos\theta\cos\phi} = \tan\theta + \tan\phi$.

(vii) $\sin(A + B)\sin(A - B) = \cos^2 B - \cos^2 A$.

6. If A and B are positive acute angles such that $\sin A = 3/5$, and sin $\sin B = 5/12$, find the values of $\sin(A + B)$ and $\cos(A - B)$.
7. If A and B are positive acute angles such that $\sin A = 1/\sqrt{5}$ and $\sin B = 1/\sqrt{10}$, show that $A + B = \pi/4$.
8. Without the use of tables, find the values of

   (i) $\cos 48° \cos 12° - \sin 48° \sin 12°$.

   (ii) $\sin 72° \cos 12° - \cos 72° \sin 12°$,

   (iii) $\sin 32° \cos 13° + \cos 32° \sin 13°$,

   (iv) $\cos 68° \cos 8° + \sin 68° \sin 8°$.

9. Prove the following identities :

   (i) $\tan(\pi/4 + \theta) = \dfrac{1 + \tan\theta}{1 - \tan\theta}$.

   (ii) $\tan(\pi/4 + \theta)\tan(\pi/4 - \theta) = 1$.

   (iii) $(\cot A - 1)(\cot B - 1) = 2$.

10. If $\tan A = p/q$, $\tan B - r/s$, and $qs \neq 0$, prove that

$$\tan (A + B) = \frac{ps + qr}{qs - pr}.$$

What happens when $qs = pr$?

11. Prove the following identities :

    (i) $\cos (n + 1) \theta \cos (n - 1) \theta + \sin (n + 1) \theta \sin (n - 1) \theta = \cos 2 \theta$.

    (ii) $\cos 4 \theta \cos \theta - \sin 4 \theta \sin \theta = \cos 3 \theta \cos 2 \theta - \sin 3 \theta \sin 2 \theta$.

    (iii) $\tan 3 \theta \tan 2 \theta \tan \theta = \tan 3 \theta - \tan 2 \theta - \tan \theta$.

    (iv) $\tan (\theta - \phi) + \tan (\phi - \psi) + \tan (\psi - \theta) = \tan (\theta - \phi) \tan (\phi - \psi) \tan (\psi - \theta)$.

12. If A, B, C be positive acute angles such that $A + B + C = \pi$, prove that $\cot B \cot C + \cot C \cot A + \cot A \cot B = 1$.

13. If A, B, C be positive angles such that $A + B + C = \pi$, prove that $\cot A/2 + \cot B/2 + \cot C/2 = \cot A/2 \cot C/2$.

14. Show that $\tan 70° = \tan 20° + 2 \tan 50°$.

15. For what values of θ and ϕ the identities in problems 5(v) and 5(vi) hold?

16. If $\theta + \phi$ is not an odd multiple of $\pi/2$ and θ, ϕ are not even multiples of $\pi/2$, then prove that

$$\tan (\theta + \phi) = \frac{\cot \theta + \cot \phi}{\cot \theta \cot \phi - 1}.$$

3.8. Formulae for Trigonometric Functions of Multiple Angles

We shall now express trigonometric functiions of 2θ, 3θ, etc., in terms of trigonometric functions of the angle θ.

Trigonometric Functions of 2θ.

Theorem .

$$\boxed{\begin{aligned} \cos 2 \theta &= \cos^2\theta - \sin^2\theta, && (1)\\ &= 1 - 2 \sin^2\theta, && (2)\\ &= 2 \cos^2\theta - 1, && (3)\\ &= (1 - \tan^2\theta)/(1 + \tan^2\theta). && (4)\end{aligned}}$$

Proof. By the addition theorem for the cosine, we have

$\cos 2 \theta = \cos (\theta + \theta) = \cos \theta \cos \theta - \sin \theta \sin \theta = \cos^2\theta - \sin^2\theta$. ......(1)

Since $\cos^2\theta = 1 - \sin^2\theta$, therefore we have from (1),

$\cos 2 \theta = \cos^2\theta - \sin^2\theta = (1 - \sin^2\theta) - \sin^2\theta = 1 - 2 \sin^2\theta$. ......(2)

Again, since $\sin^2\theta = 1 - \cos^2\theta$, therefore we may also write (1) as

$\cos 2\theta = \cos^2\theta - (1 - \cos^2\theta) = 2 \cos^2\theta - 1$. ......(3)

Finally, since $\cos^2\theta + \sin^2\theta = 1$, therefore (1) can be written as

$$\cos 2\theta = \frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta + \sin^2\theta}.$$

Dividing the numerator and the denominator of the right hand side by $\cos^2\theta$ $(\neq 0)$, we have

$$\cos 2\theta = \frac{1 - \tan^2\theta}{1 + \tan^2\theta}. \qquad \ldots\ldots (4)$$

Theorem.

$$\boxed{\begin{aligned} \sin 2\theta &= 2 \sin\theta \cos\theta, \\ &= \frac{2 \tan\theta}{1 + \tan^2\theta}, \end{aligned}} \qquad \begin{aligned} &\ldots\ldots(5) \\ &\ldots\ldots(6) \end{aligned}$$

Proof. By the addition formulae for the sine, we have

$$\begin{aligned} \sin 2\,\theta &= \sin(\theta + \theta) \\ &= \sin\theta\cos\theta + \cos\theta\sin\theta, \\ &= 2\sin\theta\cos\theta. \end{aligned} \qquad \ldots\ldots (5)$$

$$\text{Again, } \sin 2\,\theta = \frac{2\sin\theta\cos\theta}{\cos^2\theta + \sin^2\theta}.$$

Dividing the numerator and the denominator by $\cos^2\theta\ (\neq 0)$, we get

$$\sin 2\,\theta = \frac{2\tan\theta}{1 + \tan^2\theta}. \qquad \ldots\ldots(6)$$

Theorem.

$$\boxed{\tan 2\,\theta = \frac{2\tan\theta}{1 - \tan^2\theta}.} \qquad \ldots\ldots(7)$$

Proof. By the addition theorem for tangent, we have

$$\begin{aligned} \tan 2\,\theta &= \tan(\theta + \theta) \\ &= \frac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta}, \\ &= \frac{2\tan\theta}{1 - \tan^2\theta}, \end{aligned}$$

provided none of the angles involved is an odd multiple of $\pi/2$, i.e., provided neither θ nor $2\,\theta$ is an odd multiple of $\pi/2$.

Trigonometric Functions of 3θ. Trigonometric functions of $3\,\theta$ can be found in terms of trigonometric functions of θ by repeated application of the above formulae. Thus

$$\begin{aligned} \cos 3\,\theta &= \cos(2\,\theta + \theta), \\ &= \cos 2\,\theta\cos\theta - \sin 2\,\theta\sin\theta, \\ &= (\cos^2\theta - \sin^2\theta).\cos\theta - 2\sin\theta\cos\theta.\sin\theta, \\ &= \cos^3\theta - 3\sin^2\theta\cos\theta, \end{aligned}$$

Putting $1 - \cos^2\theta$ for $\sin^2\theta$ in the above equation, we get

$$\boxed{\cos 3\theta = 4\cos^3\theta - 3\cos\theta.} \qquad \text{......(8)}$$

$$\begin{aligned}\text{Again, } \sin 3\theta &= \sin(2\theta + \theta),\\ &= \sin 2\theta \cos\theta + \cos 2\theta \sin\theta,\\ &= 2\sin\theta\cos\theta . \cos\theta + (1 - 2\sin^2\theta)\sin\theta,\\ &= 2\sin\theta\cos^2\theta + (1 - 2\sin^2\theta)\sin\theta,\\ &= 2\sin\theta(1 - \sin^2\theta) + (1 - 2\sin^2\theta)\sin\theta,\\ &= 3\sin\theta - 4\sin^3\theta.\end{aligned}$$

Hence

$$\boxed{\sin 3\theta = 3\sin - 4\sin^3\theta.} \qquad \text{......(9)}$$

$$\begin{aligned}\text{Finally, } \tan 3\theta &= \tan(2\theta + \theta),\\ &= \frac{\tan 2\theta + \tan\theta}{1 - \tan 2\theta\tan\theta},\\ &= \frac{\dfrac{2\tan\theta}{1 - \tan^2\theta} + \tan\theta}{1 - \dfrac{2\tan\theta}{1 - \tan^2\theta}.\tan\theta}, \text{ by using (7)}\\ &= \frac{2\tan\theta + \tan\theta(1 - \tan^2\theta)}{(1 - \tan^2\theta) - 2\tan^2\theta},\\ &= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}.\end{aligned}$$

Hence

$$\boxed{\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}.} \qquad \text{......(10)}$$

Note. Expressions for trigonometric functions of 4θ, 5θ etc., can be found in terms of trigonometric functions of θ by repeated applications of the above formulae.

Example 3. Calculate sin 18° and cos 36°.

Solution. We have $\cos(3 \times 18°) = \cos 54° = \sin 36° = \sin(2 \times 18°)$.

Therefore $4\cos^3 18° - 3\cos 18° = 2\sin 18° \cos 18°$,

or $\cos 18° (4\cos^2 18° - 3 - 2\sin 18°) = 0$,

or $4\cos^2 18° - 3 - 2\sin 18° = 0$,

since $\cos 18° \neq 0$,

or $4(1 - \sin^2 18°) - 3 - 2\sin 18° - 0$,

or $4\sin^2 18° + 2\sin 18° - 1 = 0$,

or $(2\sin 18° + 1/2)^2 - (1/2^2) - 1 = 0$,

or $(2\sin 18° + 1/2)^2 = 5/4$,

whence $$2 \sin 18^\circ = -1/2 + \sqrt{5/2} \text{ or } -1/2 - \sqrt{5/2},$$

i.e., $$\sin 18^\circ = \frac{\sqrt{5}-1}{4} \text{ or } -\frac{\sqrt{5}+1}{4}.$$

Since $\sin 18^\circ > 0$, therefore $\sin 18^\circ \neq -\dfrac{\sqrt{5}+1}{4}$.

Hence $$\boxed{\sin 18^\circ = \frac{\sqrt{5}-1}{4}\,.}$$

Also, $$\begin{aligned}\cos 36^\circ &= 1 - 2 \sin^2 18^\circ,\\ &= 1 - 2\left(\frac{\sqrt{5}-1}{2}\right)^2\\ &= 1 - \frac{6-2\sqrt{5}}{8},\\ &= \frac{\sqrt{5}+1}{4}.\end{aligned}$$

Thus $$\boxed{\cos 36^\circ = \frac{\sqrt{5}+1}{4}\,.}$$

EXERCISES

1. Reduce each of the following expressions to a single term involving only one function of an angle :

(i) $2 \sin 13^\circ \cos 13^\circ$. (ii) $\cos^2 25^\circ - \sin^2 25^\circ$.

(iii) $\dfrac{2 \tan \pi/5}{1 - \tan^2 \pi/5}$. (iv) $\dfrac{1 - \tan^2 \pi/9}{1 + \tan^2 \pi/9}$.

(v) $\dfrac{2 \tan 31^\circ}{1 + \tan^2 31^\circ}$. (vi) $2 \cos^2 23^\circ - 1$.

(vii) $1 - 2 \sin^2 17^\circ$. (viii) $3 \sin 15^\circ - 4 \sin^3 15^\circ$.

(ix) $4 \cos^3 \pi/10 - 3 \cos \pi/10$. (x) $\dfrac{1 - 3 \tan \pi/7}{3 \tan \pi/7 - \tan^3 \pi/7}$.

2. Express $\cos 130^\circ$ in terms of $\cos 65^\circ$.

3. Express $\cos 54^\circ$ in terms of $\cos 18^\circ$.

4. Express $\cot 72^\circ$ in terms of $\cot 24^\circ$.

5. In each of the following problems, find the value of $\sin 2\theta$, $\cos 2\theta$ and $\tan 2\theta$.
   - (i) $\sin\theta = 4/5$, θ lies in the first quadrant.
   - (ii) $\cos\theta = -5/12$, θ lies in the second quadrant.
   - (iii) $\tan\theta = 24/7$, θ lies in the third quadrant.
   - (iv) $\cot\theta = -2$, θ lies in the fourth quadrant.

6. Prove each of the following identities :
   - (i) $\cot\theta - \mathrm{ton}\ \theta = 2\cot 2\theta$.
   - (ii) $\cot\theta + \tan\theta = 2\,\mathrm{cosec}\, 2\theta$.
   - (iii) $\dfrac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta$.
   - (iv) $\dfrac{\tan 2\theta}{1+\sec 2\theta} = \tan\theta$.
   - (v) $\mathrm{Cosec}\, 2\theta - \cot 2\theta = \tan\theta$.
   - (vi) $\dfrac{\sec 8\theta - 1}{\sec 4\theta - 1} = \dfrac{\tan 8\theta}{\tan 2\theta}$.
   - (vii) $\cos\theta\cos(60^\circ - \theta)\cos(60^\circ + \theta) = (1/4)\cos 3\theta$.
   - (viii) $\tan\theta + \tan(60^\circ + \theta)\tan(120^\circ + \theta) = 3\tan 3\theta$.
   - (ix) $\sin^2 72^\circ - \sin^2 60^\circ = \dfrac{\sqrt{5}-1}{8}$.
   - (x) $\cos^2 \pi/8 + \cos^2 3\pi/8 + \cos^2 5\pi/8 + \cos^2 7\pi/8 = 2$.
   - (xi) $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.

7. Obtain the formula

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta},$$

from the addition theorem for the tangent.

8. For what values of θ is the formula 10 of the Art. 3.8 valid ?

3.9. Formulae for Associated Angles

We shall now express the trigonometric functions of the angles $-\theta$, $\dfrac{\pi}{2} \pm \theta$, $\pi \pm \theta$, $\dfrac{3\pi}{2} \pm \theta$, $2\pi \pm \theta$ in terms of trigonometric functions of θ.

We have already established (in Art. 3.2) that

$$\boxed{\begin{aligned} &\cos(-\theta) = \cos\theta,\ \sin(-\theta) = -\sin\theta,\\ &\tan(-\theta) = -\tan\theta,\ \cot(-\theta) = -\cot\theta,\\ &\sec(-\theta) = \sec\theta,\ \mathrm{cosec}\,(-\theta) = -\mathrm{cosec}\,\theta, \end{aligned}} \qquad \ldots\ldots\ldots\ldots(1)$$

and (in Art. 3.3) that

$$\boxed{\begin{array}{l} \cos(\pi/2-\theta)=\sin\theta,\ \sin(\pi/2-\theta)=\cos\theta \\ \tan(\pi/2-\theta)=\cot\theta,\ \cot(\pi/2-\theta)=\tan\theta \\ \sec(\pi/2-\theta)=\csc\theta,\ \csc(\pi/2-\theta)=\sec\theta \end{array}} \qquad \ldots\ldots\ldots\ldots(2)$$

Theorem.

$$\boxed{\begin{array}{l} \cos(\pi/2+\theta)=-\sin\theta,\ \sin(\pi/2+\theta)=\cos\theta \\ \tan(\pi/2+\theta)=-\cot\theta,\ \cot(\pi/2+\theta)=\tan\theta \\ \sec(\pi/2+\theta)=-\csc\theta,\ \csc(\pi/2+\theta)=\sec\theta \end{array}} \qquad \ldots\ldots\ldots\ldots(3)$$

Proof.

$$\begin{aligned} \cos(\pi/2+\theta) &= \cos[\pi/2-(-\theta)], \\ &= \sin(-\theta), \text{ by (2)}, \\ &= -\sin\theta, \text{ by (1)}. \\ \sin(\pi/2+\theta) &= \sin[\pi/2-(-\theta)], \text{ by (2)}, \\ &= \cos(-\theta), \text{ by (2)}, \\ &= \cos\theta, \text{ by (1)}. \end{aligned}$$

The remaining formulae can be established similarly.

Theorem.

$$\boxed{\begin{array}{l} \cos(\pi-\theta)=-\cos\theta,\ \sin(\pi-\theta)=\sin\theta, \\ \tan(\pi-\theta)=-\tan\theta,\ \cot(\pi-\theta)=\cot\theta, \\ \sec(\pi-\theta)=-\sec\theta,\ \csc(\pi-\theta)=\csc\theta. \end{array}} \qquad \ldots\ldots\ldots\ldots(4)$$

Proof. By the addition formulae for the cosine and the sine, we have

$$\begin{aligned} \cos(\pi-\theta) &= \cos\pi\cos\theta+\sin\pi\sin\theta, \\ &= (-1).\cos\theta+0.\sin\theta, \\ &= -\cos\theta. \\ \cos(\pi-\theta) &= \sin\pi.\cos\theta-\cos\pi\sin\theta, \\ &= 0.\cos\theta-(-1)\sin\theta, \\ &= \sin\theta. \end{aligned}$$

Similarly we can establish the remaining formulae.

Theorem.

$$\boxed{\begin{array}{l} \cos(\pi+\theta)=-\cos\theta,\ \sin(\pi+\theta)=-\sin\theta \\ \tan(\pi+\theta)=\tan\theta,\ \cot(\pi+\theta)=+\cot\theta \\ \sec(\pi+\theta)=-\sec\theta,\ \csc(\pi+\theta)=-\csc\theta. \end{array}} \qquad \ldots\ldots\ldots\ldots(5)$$

Proof. By the addition formulae for the cosine and the sine, we have

$$\begin{aligned}\cos(\pi+\theta) &= \cos\pi\cos\theta - \sin\pi\sin\theta,\\ &= (-1)\cos\theta - 0.\sin\theta,\\ &= -\cos\theta.\end{aligned}$$

$$\sin(\pi+\theta) = \sin\pi\cos\theta + \cos\pi\sin\theta = 0.\cos\theta + (-1).\sin\theta = -\sin\theta.$$

Similarly we can establish the remaining formulae.

Theorem.

$$\boxed{\begin{aligned}&\cos\left(\frac{3\pi}{2}-\theta\right) = -\sin\theta,\ \sin\left(\frac{3\pi}{2}-\theta\right) = -\cos\theta,\\ &\tan\left(\frac{3\pi}{2}-\theta\right) = \cot\theta,\ \cot\left(\frac{3\pi}{2}-\theta\right) = \tan\theta,\\ &\sec\left(\frac{3\pi}{2}-\theta\right) = -\csc\theta,\ \csc\left(\frac{3\pi}{2}-\theta\right) = -\sec\theta.\end{aligned}} \quad \ldots\ldots\ldots\ldots(6)$$

Proof. By the addition formulae for the cosine and the sine

$$\begin{aligned}\cos\left(\frac{3\pi}{2}-\theta\right) &= \cos\frac{3\pi}{2}\cos\theta + \sin\frac{3\pi}{2}\sin\theta,\\ &= 0.\cos\theta + (-1)\sin\theta,\\ &= -\sin\theta,\\ \sin\left(\frac{3\pi}{2}-\theta\right) &= \sin\frac{3\pi}{2}\cos\theta - \cos\frac{3\pi}{2}\sin\theta,\\ &= (-1).\cos\theta - 0.\sin\theta,\\ &= -\cos\theta.\end{aligned}$$

Similarly we can establish the remaining formulae.

Theorem.

$$\boxed{\begin{aligned}&\cos\left(\frac{3\pi}{2}+\theta\right) = \sin\theta,\ \sin\left(\frac{3\pi}{2}+\theta\right) = -\cos\theta,\\ &\tan\left(\frac{3\pi}{\ }+\theta\right) = -\cot\theta,\ \cot\left(\frac{3\pi}{2}+\theta\right) = -\tan\theta,\\ &\sec\left(\frac{3\pi}{2}+\theta\right) = \operatorname{cosec}\theta,\ \operatorname{cosec}\left(\frac{3\pi}{2}+\theta\right) = -\sec\theta.\end{aligned}} \quad \ldots\ldots\ldots(7)$$

Proof.

$$
\begin{aligned}
\cos\left(\frac{3\pi}{2}+\theta\right) &= \cos\left[\frac{3\pi}{2}-(-\theta)\right] \\
&= -\sin(-\theta) \quad , \text{ by (6)} \\
&= -(-\sin\theta) \quad , \text{ by (1)} \\
&= \sin\theta.
\end{aligned}
$$

The remaining formulae can also be established similarly.

Theorem.

$$
\boxed{\begin{array}{ll}
\cos(2\pi-\theta) = \cos\theta, & \sin(2\pi-\theta) = -\sin\theta \\
\tan(2\pi-\theta) = -\tan\theta, & \cot(2\pi-\theta) = -\cot\theta \\
\sec(2\pi-\theta) = \sec\theta, & \csc(2\pi-\theta) = -\csc\theta.
\end{array}} \qquad \text{.........(8)}
$$

Proof.

$$
\begin{aligned}
\cos(2\pi-\theta) &= \cos 2\pi\cos\theta + \sin 2\pi\sin\theta, \\
&= 1.\cos\theta + 0.\sin\theta, \\
&= \cos\theta.
\end{aligned}
$$

Similarly we may prove the remaining formulae.

Theorem.

$$
\boxed{\begin{array}{ll}
\cos(2\pi+\theta) = \cos\theta, & \sin(2\pi+\theta) = \sin\theta, \\
\tan(2\pi+\theta) = \tan\theta, & \cot(2\pi+\theta) = \cot\theta, \\
\sec(2\pi+\theta) = \sec\theta, & \csc(2\pi+\theta) = \csc\theta.
\end{array}} \qquad \text{.........(9)}
$$

Proof. $2\pi + \theta$ is coterminal with θ.

Theorem.

$$
\boxed{\begin{array}{ll}
\cos(2k\pi+\theta) = \cos\theta, & \sin(2k\pi+\theta) = \sin\theta \\
\tan(2k\pi+\theta) = \tan\theta, & \cot(2k\pi+\theta) = \cot\theta \\
\sec(2k\pi+\theta) = \sec\theta, & \csc(2k\pi+\theta) = \csc\theta,
\end{array}} \qquad \text{.........(10)}
$$

where k is any integer.

Proof. If k be an integer, then the angles θ and $2k\pi + \theta$ are coterminal.

With the help of the formulae (10) we can reduce the trigonometric functions of any angle (however large or small) to those of positive angles less than 2π. By applying the formulae (3) or (5) or (7) (depending upon the angle involved) we may then reduce them to trigonometric functions of positive angles less than $\pi/2$. Finally, by applying formulae (2) we may reduce these further to trigonometric functions of positive angles not exceeding $\pi/4$.

Example. Calculate $\cos(-1020^\circ)$.

Solution.

$$\begin{aligned}\cos(-1020^\circ) &= \cos(1020^\circ),\\ &= \cos(2.360^\circ + 300^\circ),\\ &= \cos(300^\circ),\\ &= \cos(270^\circ + 30^\circ),\\ &= \sin 30^\circ,\\ &= 1/2.\end{aligned}$$

Example. Calculate $\tan 570^\circ$.

Solution.

$$\begin{aligned}\tan 570^\circ &= \tan(360^\circ + 210^\circ),\\ &= \tan 210^\circ,\\ &= \tan(180^\circ + 30^\circ),\\ &= \tan 30^\circ,\\ &= 1/\sqrt{3}.\end{aligned}$$

EXERCISES

1. Find the values of :

(i) $\cos 210^\circ$. (ii) $\sin 225^\circ$.

(iii) $\tan 330^\circ$. (iv) $\cot(-315^\circ)$.

(v) $\sec 240^\circ$. (vi) $\csc(-150^\circ)$.

(vii) $\sin 330^\circ \tan 135^\circ + \cos 225^\circ \sin 135^\circ$.

(viii) $\cos(-765^\circ)\sin(405^\circ) + \tan 300^\circ \csc 120^\circ$.

2. Express the following in terms of trigonometric functions of θ :—

(i) $\sin(270^\circ + \theta)$. (ii) $\cos(270^\circ - \theta)$.

(iii) $\tan(2\pi - \theta)$. (iv) $\cot(\pi + \theta)$.

(v) $\sec(\pi - \theta)$. (vi) $\csc(\pi/2 + \theta)$.

(vii) $\dfrac{\sin(180^\circ + \theta)\sec(-\theta)\cot(90^\circ - \theta)}{\tan(270^\circ - \theta)\cos(360^\circ - \theta)\csc(180^\circ - \theta)}$.

3. Express the following in terms of trigonometrical ratios of positive acute angles not greater than 45° :

(i) $\sin(-971^\circ)$. (ii) $\cos(567^\circ)$.

(iii) $\tan(425^\circ)$. (iv) $\cot(-762^\circ)$.

(v) $\sec(-823^\circ)$. (vi) $\csc(1146^\circ)$.

4. Find the angles between 0° and 360° which have their cosines equal to $\sqrt{3}/2$.

5. Find the angle between 0° and 360° whose sine is $-\sqrt{3}/2$ and whose cotangent is equal to $1/\sqrt{3}$.

3.10. Formulae for Trigonometric Functions of Half-Angles

Sometimes we may be required to find the values of the trigonometric functions of half an angle from the values of the trigonometric functions of the whole angle. This can be done by using the double-angle formulae of Art 3.8. In fact, from relations (2) and (3) of Art. 3.8 we have

$\cos 2\theta = 1 - 2\sin^2\theta$ and $\cos 2\theta = 2\cos^2\theta - 1$.

Replacing θ by $\theta/2$ and 2θ by θ in the above formulae we have $\cos\theta = 1 - 2\sin^2(\frac{1}{2}\theta)$ and $\cos\theta = 2\cos^2(\frac{1}{2}\theta) - 1$.

Solving these equations for sin $(\theta/2$ and cos $(\theta/2)$ respectively, we have

$$\boxed{\begin{aligned} \sin(\theta/2) &= \pm\sqrt{\left\{\frac{1-\cos\theta}{2}\right\}}, \qquad \text{...........(1)} \\ \cos(\theta/2) &= \pm\sqrt{\left\{\frac{1+\cos\theta}{2}\right\}}. \qquad \text{...........(2)} \end{aligned}}$$

Also, by dividing both sides of the identity (1) by the corresponding sides of (2) we have,

$$\boxed{\tan(\theta/2) = \pm\sqrt{\left\{\frac{1-\cos\theta}{1+\cos\theta}\right\}},} \qquad \text{...........(3)}$$

for all permissible values of θ.

The ambiguous sign on the right hand sides of (1), (2), and (3) is determined by the quadrant in which the angle $\theta/2$ lies. Thus, for example, if $\theta/2$ lies in the third quadrant, then $\sin(\theta/2)$ and $\cos(\theta/2)$ are both negative, and $\tan(\theta/2)$ is positive. Consequently, in this case we shall have to choose the negative sign before the radical in (1) and (2), and the positive sign before the radical in (3). Since the angle appearing in the left hand side of the formulae (1) to (3) is half the angle appearing in the right hand side, therefore these formulae are called the *half-angle formulae*.

Two formulae for $\tan(\theta/2)$ which may well be called half-angle formulae and which have the advantage that they do not involve any ambiguous signs (but which involve two trigonometric functions instead of one) can be obtained as follows :

$$\tan(\theta/2) = \frac{\sin(\theta/2)}{\cos(\theta/2)} = \frac{2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)} = \frac{\sin\theta}{1+\cos\theta}.$$

Hence
$$\boxed{\tan(\theta/2) = \frac{\sin\theta}{1+\cos\theta}.}$$

Similarly
$$\tan(\theta/2) = \frac{\sin(\theta/2)}{\cos(\theta/2)} = \frac{2\sin^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)} = \frac{1-\cos\theta}{\sin\theta}.$$

Hence
$$\boxed{\tan(\theta/2) = \frac{1-\cos\theta}{\sin\theta}.}$$

Example 4. Calculate sin $(\pi/8)$ and cos $(\pi/8)$.

Solution. Putting $\theta = \pi/4$ in the formulae
$$\sin(\theta/2) = \pm\sqrt{\left\{\frac{1-\cos\theta}{2}\right\}},$$
$$\cos(\theta/2) = \pm\sqrt{\left\{\frac{1+\cos\theta}{2}\right\}},$$

we have
$$\sin(\pi/8) = \pm\sqrt{\left\{\frac{1-\cos\pi/4}{2}\right\}}, \qquad \text{........ (1)}$$
$$\cos(\pi/8) = \pm\sqrt{\left\{\frac{1+\cos\pi/4}{2}\right\}}. \qquad \text{.........(2)}$$

Since $\pi/8$ lies in the first quadrant, therefore sin $(\pi/8)$ and cos $(\pi/8)$ are both positive and therefore the positive sign has to be taken before each of the radicals in (1) and (2).

Therefore
$$\sin(\pi/8) = \sqrt{\left\{\frac{1-\cos\pi/4}{2}\right\}} = \sqrt{\left\{\frac{1-1/\sqrt{2}}{2}\right\}} = \sqrt{\left\{\frac{\sqrt{2}-1}{2\sqrt{2}}\right\}}.$$
$$\cos(\pi/8) = \sqrt{\left\{\frac{1+\cos\pi/4}{2}\right\}} = \sqrt{\left\{\frac{1+1/\sqrt{2}}{2}\right\}} = \sqrt{\left\{\frac{\sqrt{2}+1}{2\sqrt{2}}\right\}}.$$

Example 5. Given sin $\theta = -5/13$ and $\pi < \theta < 3\pi/2$, calculate cos $(\theta/2)$, sin $(\theta/2)$ and tan $(\theta/2)$.

Solution. In order to apply the half-angle formulae we have to calculate first $\cos\theta$.

Now $\cos^3\theta = 1 - \sin^2\theta = 1 - (-5/13)^2 = 144/169$,

so that $\cos\theta = \pm\dfrac{12}{13}$. $\qquad$ ....(1)

Since θ lies in the third quadrant, therefore $\cos\theta$ is negative, so that the negative sign should be taken on the right hand side of (1).

Thus $\cos\theta = -12/13$.

From the half-angle formulae we have

$$\cos(\theta/2) = \pm\sqrt{\left\{\frac{1+\cos\theta}{2}\right\}} = \pm\sqrt{\left\{\frac{1-12/13}{2}\right\}} = \pm\sqrt{\left\{\frac{1}{26}\right\}} = \pm 1/\sqrt{26}, \qquad \text{.........(2)}$$

$$\sin(\theta/2) = \pm\sqrt{\left\{\frac{1-\cos\theta}{2}\right\}} = \pm\sqrt{\left\{\frac{1+12/13}{2}\right\}} = \pm 5/\sqrt{26}, \qquad \text{.........(3)}$$

$$\tan(\theta/2) = \pm\sqrt{\left\{\frac{1-\cos\theta}{1+\cos\theta}\right\}} = \pm\sqrt{\left\{\frac{1+12/13}{1-12/13}\right\}} = \pm 5. \qquad \text{.........(4)}$$

Since $\pi < \theta < 3\pi/2$, therefore $\pi/2 < \theta/2 < 3\pi/4$, i.e. $\theta/4$ lies in the second quadrant. Consequently, $\cos(\theta/2)$ is negative, $\sin(\theta/2)$ is positive and $\tan(\theta/2)$ is negative. Therefore the —ve sign should be chosen in the right-hand side of (2), the +ve sign should be chosen in the right-hand side of (3) and the —ve sign should be chosen in the right-hand side of (4).

Hence $\cos(\theta/2) = -1/\sqrt{26}$, $\sin(\theta/2) = 5/\sqrt{26}$ and $\tan(\theta/2) = -5$.

EXERCISES

1. Given $\cos 30° = \sqrt{3}/2$, deduce the values of $\sin 15°$, $\cos 15°$ and $\tan 15°$.

2. Show that $\tan 22\tfrac{1}{2}° = \sqrt{\left\{\dfrac{2-\sqrt{2}}{2+\sqrt{2}}\right\}}$.

3. Given $\cos\theta = 3/5$ and $0 < \theta < 90°$, find the values of $\sin(\theta/2)$, $\cos(\theta/2)$ and tant $(\theta/2)$.

4. Given $\cos\theta = -7/25$ and θ lies in the third quadrant, find the value of $\sin(\theta/2)$, $\cos(\theta/2)$ and tant $(\theta/2)$.

5. Given $\sin\theta = 5/13$ and θ lies in the second quadrant, find the value of $\sin(\theta/2)$, $\cos(\theta/2)$ and $\tan(\theta/2)$.

6. Given $\tan\theta = -4/3$ and θ lies in the fourth quadrant, find the values of $\sin(\theta/2)$, $\cos(\theta/2)$ and $\tan(\theta/2)$.

7. Show that

$$\text{(i)} \quad \sin\theta = \pm\sqrt{\left\{\frac{1-\cos 2\theta}{2}\right\}}.$$

$$\text{(ii)} \quad \cos\theta = \pm\sqrt{\left\{\frac{1+\cos 2\theta}{2}\right\}}.$$

8. Identify as true or false and give reasons:

$$\text{(i)} \quad \cos 100^\circ = -\sqrt{\left\{\frac{1+\cos 200^\circ}{2}\right\}}.$$

$$\text{(ii)} \quad \sin 130^\circ = \sqrt{\left\{\frac{1-\cos 260^\circ}{2}\right\}}.$$

$$\text{(iii)} \quad \tfrac{1}{2}(\cos 200^\circ) = \sqrt{\left\{\frac{1+\cos 400^\circ}{8}\right\}}.$$

$$\text{(iv)} \quad |\cos(\theta/_2)| = \sqrt{\left\{\frac{1+\cos\theta}{2}\right\}}.$$

$$\text{(v)} \quad 2\sin 300^\circ = -\sqrt{(1+\sin 60^\circ)} - \sqrt{(1-\sin 60^\circ)}.$$

3.11. Formulae for the Transformations of Products into Sums and vice-versa

We have

$$\cos(\theta-\phi) = \cos\theta\cos\phi + \sin\theta\sin\phi,$$

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi.$$

Adding and substracting in turn we get

$$\boxed{\begin{aligned} 2\cos\theta\cos\phi &= \cos(\theta-\phi) + \cos(\theta+\phi), \\ 2\sin\theta\sin\phi &= \cos(\theta-\phi) - \cos(\theta+\phi). \end{aligned}} \quad \begin{aligned} &\ldots\ldots(1) \\ &\ldots\ldots(2) \end{aligned}$$

Again from

$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi,$$

$$\sin(\theta-\phi) = \sin\theta\cos\theta - \cos\phi\sin\phi.$$

we have by addition and substraction in turn,

$$\boxed{\begin{aligned} 2\sin\theta\cos\phi &= \sin(\theta+\phi) + \sin(\theta-\phi), \\ 2\cos\theta\sin\phi &= \sin(\theta+\phi) - \sin(\theta-\phi). \end{aligned}}$$

.........(3)

.........(4)

Let us write $\theta + \phi = \alpha$ and $\theta - \phi = \beta$, so that

$$\theta = \left(\frac{\alpha+\beta}{2}\right) \text{ and } \phi = \left(\frac{\alpha-\beta}{2}\right).$$

Then formula (1) becomes

$$\boxed{\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right).} \qquad ....(5)$$

The formula (3) becomes

$$\boxed{\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right).} \qquad ....(6)$$

Also, the formula (2) becomes

$$\boxed{\cos\alpha - \cos\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\beta\alpha-\alpha}{2}\right).} \qquad ....(7)$$

Finally, the formula (4) becomes

$$\boxed{\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right).} \qquad ....(8)$$

Example 6. Prove the following identities:

$$\boxed{\begin{aligned} &\text{(i)} \quad \sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B, \\ &\text{(ii)} \quad \cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B. \end{aligned}}$$

Solution.

(i) $$\begin{aligned}\sin(A+B)\sin(A-B) &= \tfrac{1}{2}[\cos\{(A+B)-(A-B)\} \\ &\qquad - \cos\{(A+B)+(A-B)\}] \\ &= \tfrac{1}{2}[\cos 2B - \cos 2A], \\ &= \tfrac{1}{2}[(1-2\sin^2 B-(1-2\sin^2 A)], \\ &= \sin^2 A - \sin^2 B.\end{aligned}$$

(ii) $$\begin{aligned}\cos(A+B)\cos(A-B) &= \tfrac{1}{2}[\cos\{(A+B)+(A-B)\} \\ &\qquad + \cos\{(A+B)-(A-B)\}], \\ &= \tfrac{1}{2}[\cos 2A + \cos 2B], \\ &= \tfrac{1}{2}[(2\cos^2 A-1)+(1-2\sin^2 B)], \\ &= \cos^2 A - \sin^2 B.\end{aligned}$$

Example 7. If $A + B + C = 180°$, then show that
$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.$

Solution.

$$\begin{aligned}\text{L.H.S.} &= \sin 2A + \sin 2B + \sin 2C, \\ &= 2\sin(A+B)\cos(A-B) + 2\sin C\cos C, \\ &= 2\sin(\pi - C)\cos(A-B) + 2\sin C\cos[\pi-(A+B)], \\ &\quad \text{since } A+B+C=\pi, \\ &= 2\sin C\cos(A-B) - \cos(A+B)\,2\sin C, \\ &= 2\sin C[\cos(A-B)-\cos(A+B)], \\ &= 2\sin C.\,2\sin A\sin B, \\ &= 4\sin A\sin B\sin C, \\ &= \text{R.H.S.}\end{aligned}$$

Example 8. If $A + B + C$ $180°$, then show that
$\sin^2 A - \sin^2 B + \sin^2 C = 2 \sin A \cos B \sin C.$

Solution.

$$\begin{aligned}\text{L.H.S.} &= \sin^2 A - \sin^2 B + \sin^2 C, \\ &= \sin(A+B)\sin(A-B) + \sin^2 C, \\ &= \sin(\pi - C)\sin(A-B) + \sin C.\sin\{\pi-(A+B)\}, \\ &\quad \text{since } A+B+C=\pi, \\ &= \sin C\sin(A-B) + \sin C\sin A+B), \\ &= \sin C[\sin(A-B)+\sin(A+B)], \\ &= \sin C.\,2\sin A\cos B, \\ &= 2\sin A\cos B\sin C, \\ &= \text{R.H.S.}\end{aligned}$$

Example 9. If $A + B + C = 180°$, prove that

$$\cos A + \cos B - \cos C = -1 + 4 \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}.$$

Solution.

$$\begin{aligned} \text{L.H.S.} &= \cos A + \cos B - \cos C \\ &= 2 \cos \{ \tfrac{1}{2} (A + B)\} \cos \{ \tfrac{1}{2} (A - B)\} - \cos C, \\ &= 2 \cos \{ \tfrac{1}{2} (\pi - C)\} \cos \{ \tfrac{1}{2} (A - B)\} - \cos C, \end{aligned}$$

since $A + B + C = \pi$,

$$\begin{aligned} &= 2 \sin \frac{C}{2} \cos \{ \tfrac{1}{2} (A - B)\} - (1 - 2 \sin^2 \frac{C}{2}), \\ &= -1 + 2 \sin \frac{C}{2} [\cos \{ \tfrac{1}{2} (A - B)\} + \sin \frac{C}{2}], \\ &= -1 + 2 \sin \frac{C}{2} [\cos \{ \tfrac{1}{2} (A - B)\} + \sin \tfrac{1}{2} \{ \pi - (A + B)\}], \end{aligned}$$

since $A + B C = \pi$,

$$\begin{aligned} &= -1 + 2 \sin \frac{C}{2} [\cos \{ \tfrac{1}{2} (A - B)\} + \cos \{ \tfrac{1}{2} (A + B)\}], \\ &= -1 + 2 \sin \frac{C}{2} . 2 \cos \frac{A}{2} \cos \frac{B}{2}, \\ &= -1 + 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \sin \frac{C}{2}. \end{aligned}$$

EXERCISES

1. Express each of the following products as an algebraic sum of sines and cosines:

(i) $2 \sin 60° \cos 30°$.

(ii) $2 \cos 70° \sin 30°$.

(iii) $2 \cos 45° \cos 15°$.

(iv) $2 \sin 40° \sin 60°$.

(v) $2 \sin 5\theta \cos\theta$.

(vi) $\cos (A + B) \cos (A - B)$.

(vii) $4 \cos (\theta + 60°) \sin (\theta - 60°)$.

(viii) $8 \sin 2A \sin 2B$.

2. Express each of the following as product:

(i) $\sin 48^\circ + \sin 24^\circ$. (ii) $\sin 72^\circ - \sin 12^\circ$.

(iii) $\cos 5\theta + \cos 3\theta$. (iv) $\cos \frac{2\pi}{3} - \cos \frac{2\pi}{9}$.

(v) $\sin (9\theta + \phi) \sin - (2\theta - \phi)$. (vi) $\cos (3\alpha - 5\beta) - \cos (3\alpha + 5\beta)$.

3. Prove the following identities.

(i) $\dfrac{\sin 5\theta - \sin 3\theta}{\cos 3\theta - \cos 5\theta} = \cot 4\theta$.

(ii) $\dfrac{\cos 2\theta - \cos 4\theta}{\cos 2\theta + \cos 4\theta} = \dfrac{\tan\theta}{\cot 3\theta}$.

(iii) $\dfrac{\cos\theta - \cos 2\theta + \cos 3\theta}{\sin\theta - \sin 2\theta + \sin 3\theta} = \cot 2\theta$.

(iv) $\sin\theta + \sin (120^\circ + \theta) \sin (240^\circ + \theta) = 0$.

(v) $\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ = 3/16$.

(vi) $\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ = 1/16$.

4. If $A + B + C = \pi$, then prove the following identities:

(i) $\sin 2A - \sin 2B + \sin 2C = 4 \cos A \sin B \cos C$.

(ii) $\cos 2A + \cos 2B + \cos 2C = 1 - 4 \cos A \cos B \cos C$.

(iii) $\sin A + \sin B - \sin C = 4 \sin \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2}$.

(iv) $\cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$.

(v) $\sin^2 A + \sin^2 B - \sin^2 C = 2 \sin A \sin B \cos C$.

(vi) $\cos^2 A/2 + \cos^2 B/2 - \cos^2 C/2 = 2 \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}$.

5. If $\alpha + \beta + \Upsilon = \pi/2$ prove that
$\sin^2 \alpha + \sin^2 \beta + \sin^2 \Upsilon + 2 \sin \alpha \sin \beta \sin \Upsilon = 1$.

3.12. Table of Natural Trigonometric Functions

We have seen above how values of trigonometric functions of certain angles are calculated by geometric methods. But these are not enough, for in our study of trigonometry, we shall come across trigonometric functions of other angles as well. By advanced methods, the values of trigonometric functions of any angle can be computed to any desired degree of accuracy.

Table 1 at the end of the book gives the values of trigonometric functions of angles from 0° to 90° at intervals of 10′. This table is called the table of natural trigonometric functions in order to distinguish it from the table of logarithmic trigonometric functions with which we shall concern ourselves in chapter 5.

The values of trigonometric functions are, in most cases, non-terminating decimals, and therefore the values in the table are only approximations (except possibly in those cases in which the values happen to be terminating decimals). The values in the table are given to four significant figures and therefore the table is called a *four-figure table*.

Let us have a look at the table. The first column contains angles from 0° to 45° at intervals of 10′. The next six columns contain the values of the trigonometric functions of these angles, the particular function, the values of which are contained in any column is indicated at the top of the column. To find the value of any trigonometric function of an angle not exceeding 45° we first read *down* along the first column until the angle is found and then read to the *right* until we reach the column which contains the name of the function that we wish to find, at the *top*. The last column contains angles from 45° to 90° at intervals of 10′. The six columns preceding it contain the values of the trigonometric functions of these angles; the particular function, the values of which are contained in any column is indicated at the bottom of the column. To find the value of any trigonometric function of an angle lying between 45° and 90° we first read *up* along the last column until the angle is found and then read to the *left* until we reach the column which contains the name of the function that we wish to find, at the *bottom*.

The table can be used for two purposes:

1. Given an angle, to find the value of any of its trigonometric functions.
2. Given the value of a trigonometric function of an angle, to find the angle.

We shall now consider some examples to illustrate the manner in which the table can be used for any of the above purposes.

Example 10. Find

(i) $\sin 32°$;
(ii) $\tan 41° 20'$;
(iii) $\csc 25° 40'$.

Solution. (i) In table 1 at the end of the book we read down the first column until we come to 32°. Then we look across the page in the line corresponding to 32° and read from the column headed by *sin*. We thus have $\sin 32° = \cdot 5299$.

(ii) In table 1, we read down the first column until we come to 41° 21′. Then we look across the page in the line corresponding to 41° 20′ and read from the column headed by *tan*.

We thus have $\tan 41° \ 20' = \cdot 8796$.

(iii) In table 1, we read down the first column until we come to 25° 40′. Then we look across the page in the line corresponding to 25° 40′ and read from the column headed by *csc*.

We thus have $\csc 25° \ 40' = 2{\cdot}309$.

Example 11. Find

(i) $\cos 68°$.
(ii) $\cot 53° \ 10'$.
(iii) $\sec 71° 50'$.

Solution. (i) In table 1, we read up the last column until we come to 68°. Then we look across the page in the line corresponding to 68° and read from the column having *cos* at its foot.

We thus have $\cos 68° = \cdot 3746$.

(ii) In table 1, we read up the last column until we come to 53° 10′. Then we look across the page in the line corresponding to 53° 10′ and read from the column having *cot* at its foot.

We thus have $\cot 53° \ 10' = \cdot 7490$.

(iii) In table 1, we read up the last column until we come to 71° 50′ and read from the column having *sec* at its foot.

We thus have $\sec 71° \ 50' = 3{\cdot}207$.

Example 12. Find the positive acute angle θ, if

(i) $\sin \theta = \cdot 3987$;
(ii) $\tan \theta = 2{\cdot}177$;
(iii) $\sec \theta = 1{\cdot}630$.

Solution. (i) Since the sines are given in column two reading *down* and in column three reading *up*, therefore we must search for the number ·3987 through these columns. We find that ·3987 appears in column two which has *sin* at its top. This column contains the sines of angles in the first column. Reading across the page on a line with ·3987, we find in the first column the angle 23° 30′.

Hence $\sin \theta = \cdot 3987 \wedge 0 \leqslant \theta \leqslant 90° \Rightarrow \theta = 23° \ 30'$.

(ii) Since the tangents are given in column four reading *down* and in column five reading *up*, therefore we must search for the number 2·177 through these columns. We find that 2·177 appears in column five which has *tan* at its foot. This column contains the tangents of angles in the last column. Reading across the page on a line with 2·177, we find in the last column the angle 65° 20′.

Hence $\tan\theta = 2\cdot177 \wedge 0 \leqslant \theta \leqslant 90° \Rightarrow \theta = 65° \, 20'$.

(iii) Since the secants are given in column six reading *down* and in column seven reading *up*, therefore we must search for the number 1·630 through these columns. We find that 1·630 appears in column seven which has *sec* at its foot. This column contains the secants of angles in the last column. Reading across the page on a line with 1·630 we find in the last column the angle 52° 10′.

Hence $\sec\theta = 1\cdot630 \wedge 0 \leqslant \theta \leqslant 90° \Rightarrow \theta = 52° \, 10'$.

EXERCISES

1. Use table 1 (at the end of the book) to find the value of the following :

(i) sin 25° 10′. (ii) cos 48° 20′.
(iii) tan 31° 40′. (iv) cot 57° 30′.
(v) sec 36° 20′. (vi) csc 59°.
(vii) sin 74° 40′. (viii) cos 16° 10′.
(ix) tan 65° 50′. (x) cot 33° 20′.

2. Use table 1 (at the end of the book) to find the positive acute angle θ from the given value of the function :

(i) $\sin\theta = \cdot1248$ (ii) $\cos\theta = \cdot9667$
(iii) $\tan\theta = 1\cdot632$ (iv) $\cot\theta = \cdot6959$
(v) $\sec\theta = 2\cdot595$ (vi) $\csc\theta = 1\cdot228$
(vii) $\sin\theta = \cdot9261$ (viii) $\cos\theta = \cdot3529$
(ix) $\tan\theta = \cdot4986$ (x) $\cot\theta = \cdot5851$.

INTERPOLATION

In the preceding section we have seen how we can find the value of a trigonometric function of any angle which appears in the table. We shall now see how we can find the approximate value of a trigonometric function of an angle which does not appear in the table and also how we can approximately find the positive acute angle corresponding to a given functional value which does not appear in the table. The process by which this is done is called *interpolation*. Interpolation is an extremely important technique whenever we have to use tables.

In the present case it depends on the observation that small changes in the value of a trigonometric function are proportional to the changes in the angle. This can be readily seen by going through a few consecutive entries (at random) in any column of table 1. (Let us warn the reader that the above assumption is not true in general but is approximately true only for small changes.)

We shall illustrate the technique by considering a few examples.

Example 13. Find the value of $\sin 34^\circ\ 16'$.

Solution. The given angle is not listed in table 1. It lies between the angles $34^\circ\ 10'$ and $34^\circ\ 20'$ which happen to be consecutive entries in the table. From the table we can see that $\sin\theta$ increases as θ increases from 0° to 90°. Therefore

$$\sin 34^\circ 10' < \sin 34^\circ 16' < \sin 34^\circ 20'.$$

From the table, we find

$$\begin{aligned} \sin 34^\circ\ 10' &= \cdot 5616 \\ \sin 34^\circ\ 16' &= ? \\ \text{and } \sin 34^\circ\ 20' &= \cdot 5640. \end{aligned}$$

As the angle increases by $10'$, its sine increases by 24 thousandths. Now the angle $34^\circ\ 16'$ is $6/10$ of the way from $34^\circ\ 10'$ to $34^\circ\ 20'$. Therefore $\sin 34^\circ\ 16'$ is also $6/10$ of the way from $\cdot 5616$ to $\cdot 5640$. Now $\{\frac{6}{10} \text{ of}\}\ (24) = 14\frac{2}{5} = 14$. (We have rounded off $14\frac{2}{5}$ to 14 because $14\frac{2}{5}$ is closer to 14 than 15). Since the sine is *increasing*, therefore we must *add* (to $\cdot 5616$, so as to have 14 thousandths). Thus

$$\sin 34^\circ\ 16' = \cdot 5630.$$

Remark. The difference $\cdot 0024$ between two consecutive entries $\cdot 5616$ and $\cdot 5640$ is called the *tabular difference*. The number 14 thousandths is called the *correction*. The correction is to be added in the case of sin, tan and sec, because they are increasing functions as θ increases from 0° to 90°. Since the functions cos, cot and csc are decreasing functions as θ increases from 0° to 90°, therefore the correction has to be subtracted from the functional value of the smaller of the two consecutive entries between which the angle in question lies.

Example 14. Find $\cos 69^\circ\ 13'$.

Solution. The given angle, $69^\circ\ 13'$ is not listed in table 1. It lies between the consecutive entries $69^\circ\ 10'$ and $69^\circ\ 20'$ in the table.

Now

$$\begin{aligned} \cos 69^\circ\ 10' &= \cdot 3557 \\ \cos 69^\circ\ 13' &= ? \\ \cos 69^\circ\ 20' &= \cdot 3529. \end{aligned}$$

The tabular difference is ·0028. Since the given angle is 3/10 of the way from 69° 10′ to 69° 20′, therefore the correction is $\cdot 0028 \times (3/10) = \cdot 0008$. Since cos θ increases from 0° to 90°, therefore the correction has to be subtracted from ·3557, so as to give

$$\cos 69^\circ\ 13' = \cdot 3557 - \cdot 0008 = \cdot 3549.$$

Example 15. Find the positive acute angle for θ which $\tan\theta = 1{\cdot}689$.

Solution. Since the tangents are given in column four reading *down* and in column five reading *up,* therefore we must search for the number 1·689 through these columns but there are two consecutive entries 1·686 and 1·698 in column six between which the given number lies. Since the entries in column five give tangents of angles in the last column, therefore reading the entries in the last column against the above entries, we have

$$\begin{aligned} \tan 59^\circ\ 20' &= 1{\cdot}686 \\ \tan\ \ ? \quad &= 1{\cdot}689 \\ \tan 59^\circ\ 30' &= 1{\cdot}698 \end{aligned}$$

Since 1·689 is 3/12 of the way from 1·686 to 1·689, and since the tangent of an angle increases as θ increases from 0° to 90°, therefore we must add $3/12 \times 10' = 3'$ (rounded off to the nearest minute) to 59° 20′.

We thus find that tan 59° 23′ = 1·689.

Hence the required angle is 59° 23′.

Remark. After some practice, the reader should inculcate the habit of performing the interpolation mentally.

EXERCISES

1. Find the value of the following (use table 1 and interpolate):

(i) sin 28° 23′.
(ii) cos 51° 47′.
(iii) tan 39° 56′.
(iv) cot 48° 38′.
(v) sec 66° 25′.
(vi) csc 78° 42′.
(vii) sin 46° 53′.
(viii) cos 33° 46′.
(ix) tan 56° 37′.
(x) cot 41° 29′.
(xi) sec 15° 24′.
(xii) csc 35° 44′.

2. Find, to the nearest minute, the positive acute angle θ, for which:

(i) $\sin\theta = \cdot 0627$ (ii) $\cos\theta = \cdot 5380$
(iii) $\tan\theta = \cdot 4329$ (iv) $\cot\theta = 1\cdot 636$
(v) $\sec\theta = 2\cdot 537$ (vi) $\csc\theta = 1\cdot 395$
(vii) $\sin\theta = \cdot 9479$ (viii) $\cos\theta = \cdot 4052$
(ix) $\tan\theta = \cdot 5682$ (x) $\cot\theta = \cdot 4162$
(xi) $\sec\theta = 1\cdot 823$ (xii) $\csc\theta = 2\cdot 184$

ASSORTED EXERCISES ON CHAPTER 3

1. Show that $\tan 75^\circ - \tan 30^\circ - \tan 75^\circ \tan 30^\circ = 1$.
2. Show that $\sin 10^\circ \sin 50^\circ \sin 70^\circ = 1/8$.
3. Prove that $\tan 20^\circ + \tan 80^\circ + \tan 140^\circ = 3/\sqrt{3}$.
4. Prove that $\cos 6^\circ \cos 42^\circ \cos 66^\circ \cos 78^\circ = 1/16$.
5. Show that $\sin^2 24^\circ - \sin^2 6^\circ = \dfrac{\sqrt{5}-1}{8}$.
6. Prove the following identities:

(i) $\dfrac{\tan(\pi/4+\theta)+\tan(\pi/4-\theta)}{\tan(\pi/4+\theta)-\tan(\pi/4-\theta)} = \operatorname{cosec} 2\theta$.

(ii) $\cos\beta\cos(2\alpha-\beta) = \cos^2\alpha - \sin^2(\alpha-\beta)$.

(iii) $\sin \sin(60^\circ-\theta)\sin(60^\circ+\theta) = \frac{1}{4}\sin 3\theta$.

(iv) $\tan\theta\tan(60^\circ-\theta)\tan(60^\circ+\theta) = \tan 3\theta$.

(v) $\sin\theta + \sin(120^\circ+\theta) + \sin(240^\circ+\theta) = 0$.

(vi) $\tan(\theta+30^\circ) + \cot(\theta-30^\circ) = \dfrac{1}{\sin 2\theta - \sin 60^\circ}$.

(vii) $\cos^8\theta - \sin^8\theta = \cos 2\theta - \dfrac{1}{4}\sin 2\theta \sin 4\theta$.

(viii) $\cos^3\theta + \cos^3(120^\circ+\theta) + \cos^3(240^\circ+\theta) = \dfrac{3}{4}\cos 3\theta$.

(ix) Find $\sin 9^\circ$ and $\cos 9^\circ$ by taking $\sin 18^\circ = \dfrac{\sqrt{5}-1}{4}$.

(x) Verify that (i) $\tan 7\frac{1}{2}^\circ = \sqrt{6}-\sqrt{3}+\sqrt{2}-2$.

(ii) $\cot 7\frac{1}{2}^\circ = \sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}$.

(xi) If $\cos\theta = \frac{1}{2}(a + 1/a)$, show that $\cos 3\theta = \frac{1}{2}\left(a^3 + \frac{1}{a^3}\right)$

(xii) If $\tan\beta = \frac{n \sin\alpha \cos\alpha}{1 - n^2 \sin^2\alpha}$, prove that $\tan(\alpha - \beta) = (1 - n)\tan\alpha$

(xiii) If $\alpha + \beta = \Upsilon$, prove that

$$\cos^2\alpha + \cos^2\beta + 2\cos\alpha\cos\beta\cos\Upsilon = \sin^2\Upsilon$$

(xiv) If $A + B + C = \pi$, then prove that

$$\sin A/2 + \sin B/2 + \sin C/2 = 1 + 4 \sin\frac{\pi - A}{4} \sin\frac{\pi - B}{4} \sin\frac{\pi - C}{4}$$

(xv) If $\tan\theta/2 = \sqrt{\frac{1+e}{1-e}} \tan\phi/2$, prove that $\cos\phi = \frac{\cos\theta + e}{1 + e\cos\theta}$,

$$\sin\phi = \frac{\sqrt{1 - e^2}\sin\theta}{1 + e\cos\theta}.$$

Chapter 4

Properties Of Triangles

4.1. Introduction

We shall apply the theory of trigonometric functions to discuss the properties of triangles, i.e., the relations between sides, angles, circum-radius, in-radius, ex-radii, etc. of a triangle.

Let ABC be a triangle. By the angle A we shall mean that angle between the sides AB and AC which lies between 0° and and 180°. The angles B and C are similarly defined. thus we shall always have $A > 0$, $B > 0$, $C > 0$ and $A + B + C = 180^\circ$. The sides BC, CA, AB opposites to the vertices A, B, C will be denoted by a, b, c respectively.

Thus we shall have

$$0 < a < b + c,$$
$$0 < b < c + a,$$
$$0 < c < a + b,$$

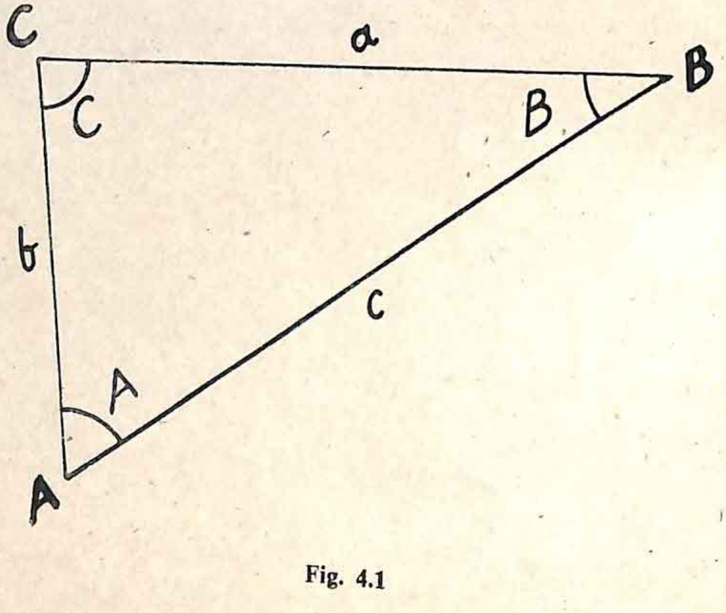

Fig. 4.1

The lengths of the radii of the circumscribed circle (or the circum-circle), the inscribed circle (or the in-circle) and the escribed circles (or the ex-circles) opposite to A, B, C will be denoted by r, R, r_1, r_2, r_3 and the centre of these circles by **O**, **I**, I_1, I_2, I_3. The area of the triangle will be denoted by $\triangle$ and the semi-perimeter by s, so that $2s = a + b + c$.

4.2. Relations between Sides and Angles of a Triangle

Theorem (The Law of Sines). In any triangle sides are proportional to the sines of opposite angles :

$$\boxed{\frac{a}{\text{Sin A}} = \frac{b}{\text{Sin B}} = \frac{c}{\text{Sin C}}} \qquad (1)$$

Proof. Let us draw the circum-circle of the triangle ABC [Figs. 4.2 (a) 4.2 (b) 4.2(c)]. Let us consider any vertex, say, A. We shall consider three cases according as the angle A is

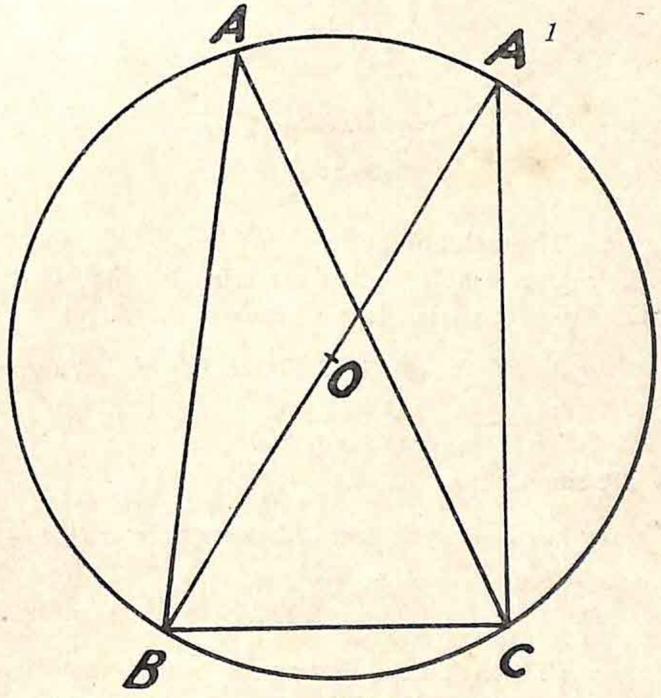

Fig. 4.2 (a)

(i) an acute angle, [Fig. 4.2(a)]
(ii) a right angle, [Fig. 4.2(b)]
and (iii) an obtuse angle. [Fig. 4.2(c)]

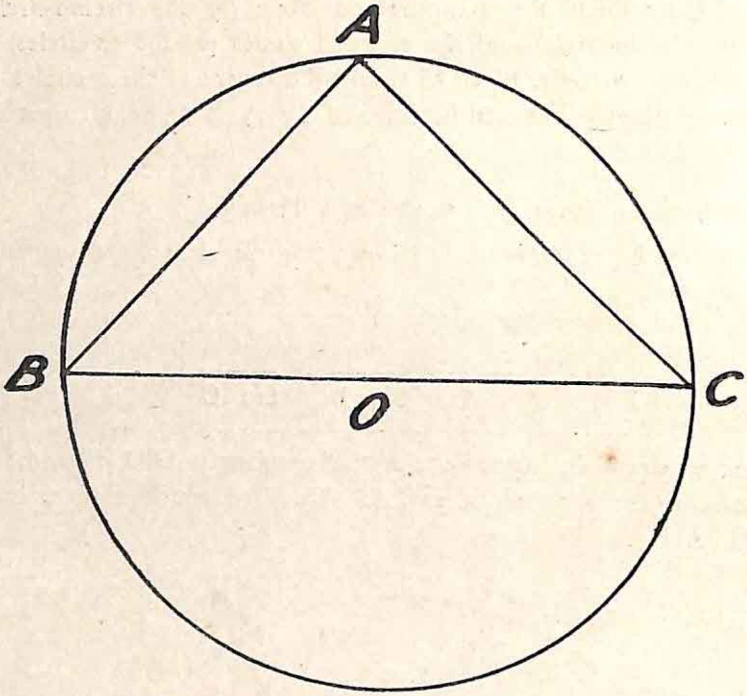

Fig. 4.2 (b)

Case (i) : A is acute. Through any other vertex, say B, draw the diameter BA′ of the circumscribed circle [Fig. 4.2(a)]. Then the triangle A′BC is right-angled with the right angle at C. From this triangle we obtain a = 2R Sin A′.

In this case the vertices A and A′ are on the same side of BC and hence

$$\angle A' = \angle A,$$

so that

$$\sin A' = \sin A,$$

and therefore a = 2R sin A.

Case (ii) : A is a right angle. In this case BC is a diameter [Fig. 4.2(b)]. We have

$$a = 2R,$$

$$= 2R \text{ Sin } A, \text{ as } \sin A = \sin 90^\circ = 1.$$

Case (iii) : A is obtuse. Through any other vertex, say B, draw the diameter BA′ of the circumscribed circle [Fig. 4.2(c)]. Then the triangle A′BC is right-angled with the right angle at C. From this triangle we obtain

$$a = 2R \sin A'.$$

In this case the vertices A and A′ are on the opposite sides of BC and hence

$$\angle A + \angle A' = 180°,$$

so that

$$\sin A' = \sin (180° - A) = \sin A,$$

and therefore $a = 2R \sin A$.

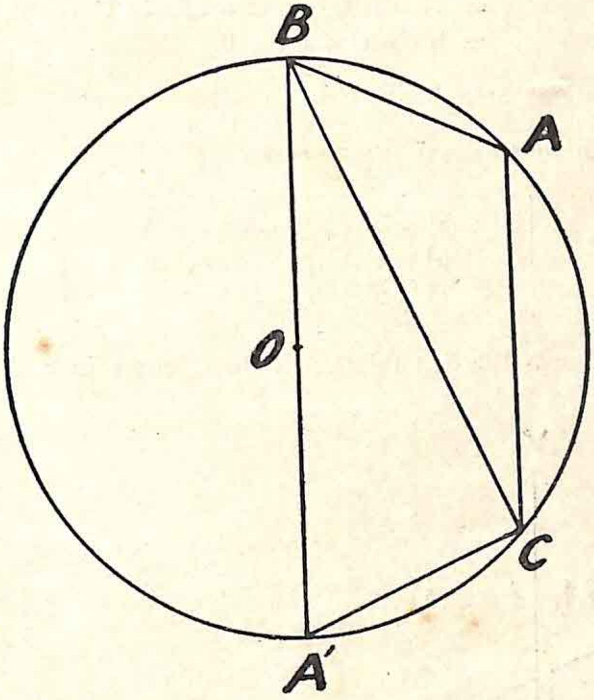

Fig. 4.2 (c)

Thus in all the three cases

$$a = 2R \sin A. \tag{2}$$

Similar equations can be obtained for other sides. Thus

$$a = 2R \sin A, \quad b = 2R \sin B, \quad c = 2R \sin C,$$

which give

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R. \tag{3}$$

Corollary. The circum-radius,

$$R = \frac{a}{2 \sin A} = \frac{b}{2 \sin B} = \frac{c}{2 \sin C}. \tag{3a}$$

Example 1. Show that

$$\boxed{\begin{aligned} a &= b \cos C + c \cos B \\ b &= c \cos A + a \cos C \\ c &= a \cos B + b \cos A. \end{aligned}} \tag{4}$$

Proof. We have

$$\begin{aligned} a &= 2R \sin A, \\ &= 2R \sin (B + C), \text{ as } A + B + C = 180°, \\ &= 2R \sin B. \cos C + 2R \sin C. \cos B, \\ &= b \cos C + c \cos B. \end{aligned}$$

The other two relations are obtained in a similar manner.

Theorem (The Law of Cosines). In any triangle

$$\boxed{\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cos A \\ b^2 &= c^2 + a^2 - 2ca \cos B \\ c^2 &= a^2 + b^2 - 2ab \cos C. \end{aligned}} \qquad (5)$$

We shall prove only the first relation. Others follow in a similar manner.

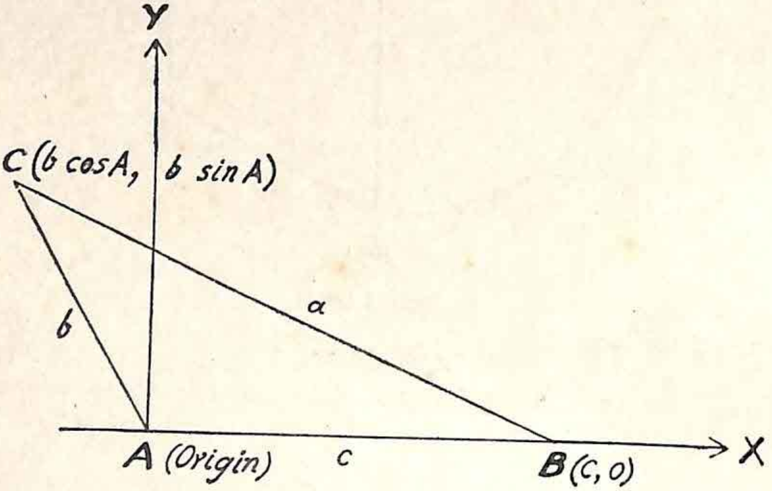

Fig. 4.3

Proof. Let us suppose that the origin of a rectangular coordinate system is at A and AB is along the positive x-axis (Fig. 4.3).

Then AC is the terminal side of the angle BAC = ∠ A. AC is of length b. B has the coordinates (c, o) and C has the coordinates (b cos A, b sin A). Therefore by Art. 1.9

$$\begin{aligned} a^2 = BC^2 &= (b \cos A - c)^2 + (b \sin A - o)^2, \\ &= b^2 (\cos^2 A + \sin^2 A) + c^2 - 2bc \cos A, \\ &= b^2 + c^2 - 2bc \cos A. \end{aligned}$$

Corollary. We have immediately from (5)

$$\boxed{\begin{aligned} \cos A &= \frac{b^2 + c^2 - a^2}{2bc} \\ \cos B &= \frac{c^2 + a^2 - b^2}{2ca} \\ \cos C &= \frac{a^2 + b^2 - c^2}{2ab.} . \end{aligned}} \tag{6}$$

Theorem (The Law of Tangents). In any triangle ABC

$$\boxed{\frac{\tan \{1/2\ (B - C)\}}{\tan \{1/2\ (B + C)\}} = \frac{b - c}{b + c}} \tag{7}$$

and two similar relations.

Proof. We have

$$\begin{aligned} \frac{b - c}{b + c} &= \frac{2R\ (\sin B - \sin C)}{2R\ (\sin B + \sin C)}, \\ &= \frac{2 \sin \{\frac{1}{2} (B - C)\} \cos \{\frac{1}{2} (B + C)\}}{2 \sin \{\frac{1}{2} (B + C)\} \cos \{\frac{1}{2} (B - C)\}}, \\ &= \frac{\tan \{\frac{1}{2} (B - C)\}}{\tan \{\frac{1}{2} (B + C)\}} . \end{aligned}$$

Corollary. $\tan \{\frac{1}{2} (B - C)\} = \frac{b - c}{b + c} \cdot \cot (A/2)$,

for $\tan \{\frac{1}{2} (B + C)\} = \tan \{\frac{1}{2} (180^\circ - A)\} = \tan (90^\circ - \frac{1}{2} A) = \cot \frac{1}{2} A$.

Example 2. Prove that in any triangle ABC,

$$\text{(i)} \quad \frac{b - c}{a} = \frac{\sin \{\frac{1}{2} (B - C)\}}{\cos (\frac{1}{2} A)} ;$$

$$\text{(ii)} \quad \frac{b + c}{a} = \frac{\cos \{\frac{1}{2} (B - C)\}}{\sin (\frac{1}{2} A)} .$$

Solution.

(i) We have

$$\begin{aligned} \frac{b - c}{a} &= \frac{2R\ (\sin B - \sin C)}{2R \sin A}, \\ &= \frac{\sin B - \sin C}{\sin A}, \end{aligned}$$

$$= \frac{2\cos\{\frac{1}{2}(B+C)\}\sin\{\frac{1}{2}(B-C)\}}{2\sin(\frac{1}{2}A)\cos(\frac{1}{2}A)},$$

$$= \frac{\sin\{\frac{1}{2}(B-C)\}}{\cos(\frac{1}{2}A)}, \text{ since } \cos\tfrac{1}{2}(B+C) = \sin(\tfrac{1}{2}A).$$

(ii) We have

$$\frac{b+c}{a} = \frac{2R(\sin B + \sin C)}{2R\sin A},$$

$$= \frac{\sin B + \sin C}{\sin A},$$

$$= \frac{2\sin\{\frac{1}{2}(B+C)\}\cos\{\frac{1}{2}(B-C)\}}{2\sin(\frac{1}{2}A)\cos(\frac{1}{2}A)},$$

$$= \frac{\cos\{\frac{1}{2}(B-C)\}}{\sin(\frac{1}{2}A)}, \quad \text{since } \sin\tfrac{1}{2}(B+C) = \cos\tfrac{1}{2}A.$$

Note. The above formulae are very useful in checking solutions of triangles, since all six parts of a triangle appear in each formula.

Half-angle Formulae. We have

$$\cos A = \frac{b^2+c^2-a^2}{2bc} \text{ and } \cos A = 2\cos^2(A/2) - 1.$$

Hence

$$2\cos^2\tfrac{1}{2}A = 1 + \frac{b^2+c^2-a^2}{2bc}$$

$$= \frac{(b+c)^2-a^2}{2bc},$$

$$= \frac{(b+c+a)\ (b+c-a)}{2bc}$$

$$= \frac{2s.\,2(s-a)}{2bc}.$$

Therefore

$$\cos(A/2) = \pm\sqrt{\left\{\frac{s(s-a)}{bc}\right\}}.$$

Since $0 < A < 180°$, therefore $0 < A/2 < 90°$, and consequently $\cos(A/2)$ is positive. We must, therefore, take the positive sign before the radical.

$$\text{Hence } \cos(A/2) = \sqrt{\left\{\frac{s(s-a)}{bc}\right\}}. \qquad (8a)$$

Similarly, it can be shown that

$$\cos (B/2) = \sqrt{\left\{\frac{s(s-b)}{ca}\right\}}, \tag{8b}$$

$$\text{and } \cos (C/2) = \sqrt{\left\{\frac{s(s-c)}{ab}\right\}}. \tag{8c}$$

Again, from

$$\cos A = 1 - 2\sin^2 (\tfrac{1}{2} A),$$

we get

$$\begin{aligned} 2\sin^2(\tfrac{1}{2}A) &= 1 - \frac{b^2+c^2-a^2}{2bc} \\ &= \frac{a^2-(b^2-c)^2}{2bc} \\ &= \frac{(a-b+c)\ (a+b-c)}{2bc} \\ &= \frac{2(s-b)\,2(s-c)}{2bc}. \end{aligned}$$

Therefore

$$\sin (A/2) = \pm \sqrt{\left\{\frac{(s-b)\ (s-c)}{bc}\right\}}.$$

Since $0 < A < 180°$, therefore $0 < A/2 < 90°$ and consequently $\sin (A/2)$ is positive. We must, therefore, take the positive sign before the radical.

$$\text{Hence } \sin (A/2) = \sqrt{\left\{\frac{(s-b)\ (s-c)}{bc}\right\}}. \tag{9a}$$

Similarly

$$\sin (B/2) = \sqrt{\left\{\frac{(s-c)\ (s-a)}{ca}\right\}} \tag{9b}$$

and

$$\sin (C/2) = \sqrt{\left\{\frac{(s-a)\ (s-b)}{ab}\right\}}. \tag{9c}$$

From formulae (8a, 8b, 8c) and (9a, 9b, 9c) we obtain, by division

$$\boxed{\begin{aligned} \tan (A/2) &= \sqrt{\left\{\frac{(s-b)(s-c)}{s(s-a)}\right\}} \\ \tan (B/2) &= \sqrt{\left\{\frac{(s-c)(s-a)}{s(s-b)}\right\}} \\ \tan (C/2) &= \sqrt{\left\{\frac{(s-a)(s-b)}{s(s-c)}\right\}} \end{aligned}} \tag{10}$$

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Corollary : $\sin A = \frac{2}{bc}\sqrt{\left\{ s(s-a)(s-b)(s-c) \right\}}$ (11)

and two similar relations.

We have

$$\sin A = 2 \sin (A/2) \cos (A/2),$$

$$= 2. \sqrt{\left\{\frac{(s-b)(s-c)}{bc}\right\}} \cdot \sqrt{\left\{\frac{s(s-a)}{bc}\right\}}, \text{ by (8a) and (9a)}$$

$$= \frac{2}{bc} \sqrt{\left\{ s(s-a)(s-b)(s-c) \right\}}.$$

Expressions for sin B and sin C are obtained in a similar manner.

Example 3. If the sides of a triangle are in A. P., prove that the cotangents of half the angles are also in A. P.

Solution. We are given that

$$a + c = 2b,$$
$$\text{i.e., } 2s - b = 2b, \qquad \text{(i)}$$

and we are required to prove that

$$\cot (A/2) + \cot (C/2) = 2 \cot (B/2). \qquad \text{(ii)}$$

Now, $\cot A/2 + \cot C/2 = \frac{\cos (A/2) \sin (C/2) + \cos (C/2) \sin (A/2)}{\sin (A/2) \sin (C/2)}$,

$$= \frac{\sin \left(\frac{A + C}{2}\right)}{\sin (A/2) \sin (C/2)},$$

$$= \frac{\cos (B/2)}{\sin (A/2) \sin (C/2)},$$

$$= 2 \cot (B/2) \left(\frac{1}{2} \cdot \frac{\sin B/2}{\sin (A/2) \sin (C/2)}\right). \qquad \text{(iii)}$$

Again, $\frac{1}{2} \cdot \frac{\sin (B/2)}{\sin (A/2) \sin C/2)} = \frac{1}{2} \cdot \sqrt{\left\{\frac{(s-c)(s-a)}{ca}\right\}} \cdot \sqrt{\left\{\frac{bc}{(s-b)(s-c)}\right\}}$

$$\sqrt{\left\{\frac{ab}{(s-a)(s-b)}\right\}},$$

$$= \frac{1}{2} \sqrt{\left\{\frac{b^2}{(s-b)^2}\right\}} = 1, \text{ by (i).}$$

Therefore, from (iii),

$$\cot (A/2) + \cot (C/2) = 2 \cot (B/2),$$

i.e., $\cot (A/2)$, $\cot (B/2)$, $\cot (C/2)$ are in A. P.

EXERCISES

In any triangle ABC, prove that

1. $a \cos A + b \cos B + c \cos C = 4R \sin A \sin B \sin C.$
2. $a \sin A - b \sin B = c \sin (A - B).$
3. $a^2 \sin (B - C) = (b^2 - c^2) \sin A.$
4. $a \cos A + b \cos B + c \cos C = 2a \sin B \sin C.$
5. $a \sin (B - C) + b \sin (C - A) + c \sin (A - B) = 0.$
6. $c = a \cos B + b \cos A.$
7. $\dfrac{c - b \cos A}{b - c \cos A} = \dfrac{\cos B}{\cos C}.$
8. $2 (bc \cos A - c a \cos B + a b \cos C) = a^2 + b^2 + c^2.$
9. $(b^2 + c^2 - a^2) \tan A = (c^2 + a^2 - b^2) \tan B.$
10. $a^2 = (b - c)^2 \cos^2 (A/2) + (b + c)^2 \sin^2 (A/2).$
11. $c^2 = (a + b)^2 - 4a b \cos^2 (C/2).$
12. $(b^2 - c^2) \cot A + (c^2 - a^2) \cot B + (a^2 - b^2) \cot C = 0.$
13. If $C = 60°$, then
$$\frac{1}{a + c} + \frac{1}{b + c} = \frac{3}{a + b + c}.$$
14. $\tan \{(1/2) (A - B)\} = \dfrac{a - b}{a + b} \cot (C/2).$
15. $\dfrac{c + a}{c - a} = \tan \dfrac{C + A}{2} \cot \dfrac{C - A}{2}.$
16. $\sin (B/2) = \sqrt{\left\{\dfrac{(s - c)\ (s - a)}{ca}\right\}}.$
17. $\sin (C/2) = \sqrt{\left\{\dfrac{(s - a)\ (s - b)}{ab}\right\}}.$
18. $\cot (A/2) + \cot (B/2) + \cot (C/2) = \dfrac{a + b + c}{a + b - c} \cot (C/2).$
19. $(b - c) \cot (A/2) + (c - a) \cot (B/2) + (a - b) \cot (C/2) = 0.$

20. $\dfrac{a+b+c}{a-b+c} = \cot(A/2)\cot(C/2)$.

21. $(b+c-a)\{\cot(B/2)+\cot(C/2)\} = 2a\cot(A/2)$.

22. If $\cot(C/2) = \dfrac{a+b}{c}$, the triangle is right-angled.

23. If $a\cos A = b\cos B$, then the triangle is either isosceles or right-angled.

24. If $b + c = 3a$, then $\cot(B/2)\cot(C/2) = 2$.

25. If $3\tan(A/2)\tan(C/2) = 1$, then a,b,c, are in A.P.

26. Deduce the formulae (5) of Art. 4.2 from the formulae (4) of Art. 4.2.

4.3. Area of a Triangle

The area $\triangle$ of a triangle ABC $= (1/2)$ base × altitude.

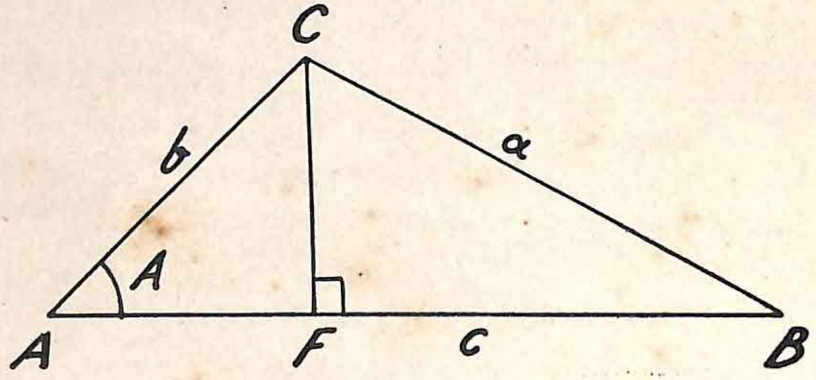


Fig. 4.4

In the triangle ABC, there must be at least two acute angles. Let us suppose that A and B are the acute angles. Let us draw the perpendicular CF from C to AB (Fig. 4.4).

$$\begin{aligned}\text{Then the altitude (to the base AB)}\\ &= CF,\\ &= b\sin A.\\ \text{Therefore } \triangle &= (1/2)\, c \times b\sin A,\\ &= (1/2)\, bc\sin A.\end{aligned}$$

Now we have from the Law of Sines [Art. 4.2 (1)]

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$

whence

$$bc \sin A = ca \sin B = ab \sin C.$$

Thus $\triangle = (1/2)\ bc \sin A = (1/2)\ ca \sin B = (1/2)\ ab \sin C,$ (i)

so that the area of a triangle is equal to half the rectangle contained by two adjacent sides multiplied by the sine of the included angle.

Hence, from (11)

$$\boxed{\triangle = \sqrt{\{ s(s-a)(s-b)(s-c)\}}.} \qquad (2)$$

Example 4. Show that in any triangle ABC,

$$\triangle = \frac{a^2 \sin B \sin C}{2 \sin (B+C)}.$$

Use the above result to show that if $B = 45°$, $C = 60°$, and $a = 2(\sqrt{3}+1)$ cm., then the area of the triangle ABC is $6 + 2\sqrt{3}$ sq. cm.

Solution. Since $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$,

therefore $b = \dfrac{a \sin B}{\sin A}$ and $c = \dfrac{a \sin C}{\sin A}$.

Hence

$$\begin{aligned}\triangle &= (1/2)\ bc \sin A,\\ &= (1/2)\ \frac{a \sin B}{\sin A} \cdot \frac{a \sin C}{\sin A} \cdot \sin A,\\ &= \frac{a^2 \sin B \sin C}{2 \sin A},\\ &= \frac{a^2 \sin B \sin C}{2 \sin (B+C)}. \qquad (1)\end{aligned}$$

Substituting $B = 45°$, $C = 60°$, and $a = 2(\sqrt{3}+1)$ cm, in (i), we have

$$\begin{aligned}\triangle &= \frac{\{2(\sqrt{3}+1)\}^2 \sin 45° \sin 60°}{2 \sin (45° + 60°)} \text{ sq. cm},\\ &= \frac{4(\sqrt{3}+1)^2 \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2}}{2\left(\frac{1}{\sqrt{2}} \cdot \frac{1}{2} + \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2}\right)} \text{ sq. cm},\\ &= 2\sqrt{3}(\sqrt{3}+1) \text{ sq. cm},\\ &= 6 + 2\sqrt{3} \text{ sq. cm}.\end{aligned}$$

Example 5. In any triangle ABC, prove that

$$\triangle = 2R^2 \sin A \sin B \sin C.$$

Solution. Since $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R,$

therefore $a = 2R \sin A,\ b = 2R \sin B,\ c = 2R \sin C.$

Now
$$\begin{aligned} \Delta &= (1/2)\, bc \sin A, \\ &= (1/2)\, 2R \sin B.\, 2R \sin C.\, \sin A, \\ &= 2R^2 \sin A \sin B \sin C. \end{aligned}$$

EXERCISES

1. Find the area of the triangle ABC when

   (i) $a = 5$ cm, $b = 12$ cm, and $c = 13$ cm.

   (ii) $a = 14$ cm, $b = 48$ cm, and $c = 50$ cm.

   (iii) $a = 45$ cm, $b = 108$ cm, and $c = 117$ cm.

   (iv) $a = \sqrt{3}$ m, $b = \sqrt{2}$m, and $c = \frac{\sqrt{6} + \sqrt{2}}{2}$ m.

   (v) $a = 10$ cm, $b = 12$ cm, and $C = 30°$.

2. If one angle of a triangle be 60°, the area be $10\sqrt{3}$ sq.cm, and the length of one of the sides containing the angle be 8 cm, find the length of the other side.

3. In any triangle ABC, prove that

   (i) $s\,(s - c) \tan (C/2) = \Delta$.

   (ii) $4\,\Delta \cot A = b^2 + c^2 - a^2$.

   (iii) $\Delta = \frac{c^2 \sin A \sin B}{2 \sin (A + B)}$.

   (iv) $abc.\, s.\, \sin (A/2) \sin (B/2) \sin (C/2) = \Delta^2$.

4. If P and Q are positive acute angles such that $P > Q$, deduce from formula 1 of art 4.3, the following formulae :

   (i) $\sin (P + Q) = \sin P \cos Q + \cos P \sin Q$;

   (ii) $\sin (P - Q) = \sin P \cos Q - \cos P \sin Q$.

4.4 Radii of the Circumscribed, Inscribed and Escribed Circles

(1) *Circum-circle :* We have from Art. 4.2 (3a),

$$R = \frac{a}{2 \sin A} = \frac{abc}{2bc \sin A},$$

and from Art. 4.3,

$$\Delta = (1/2)\, bc \sin A.$$

$$\text{Therefore} \quad \boxed{R = \frac{abc}{4\,\triangle}\,.} \tag{1}$$

(2) *In-circle :* Let AI, BI, CI (Fig. 4.5), the bisectors of the angles of the angles of the triangle ABC, meet in I, the centre of the in-circle. Draw IX, IY, IZ perpendiculars to BC, CA, AB respectively (X, Y, Z are the points of contact of the in-circles with the sides BC, CA, AB respectively); each perpendicular is of length r.

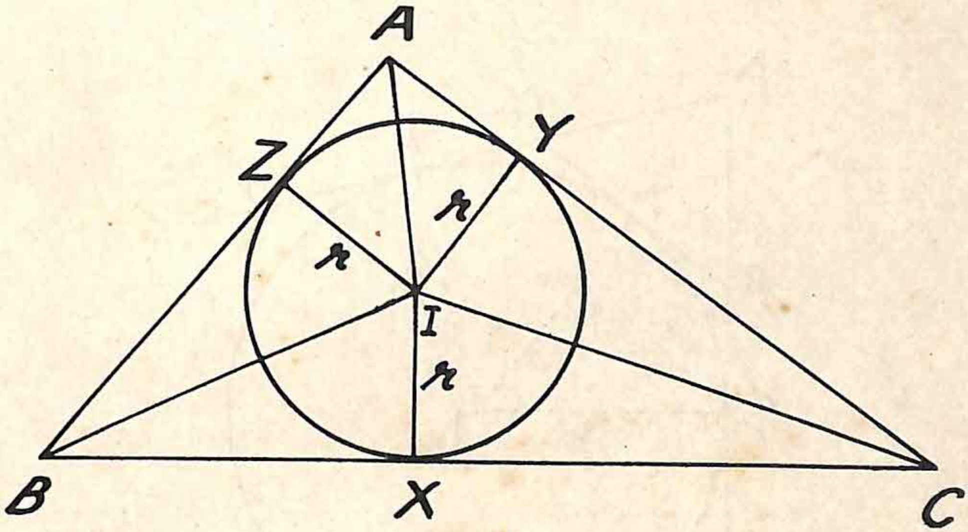


Fig. 4.5

$$\begin{aligned} \text{Then } \triangle &= IBC + \triangle ICA + \triangle\ IAB, \\ &= (1/2)\ BC.\ IX + (1/2)\ CA.\ IY + (1/2)\ AB.\ IZ, \\ &= (1/2)r.\ (a+b+c) = rs. \end{aligned}$$

$$\text{Therefore} \quad \boxed{r = \frac{\triangle}{s}\,.} \tag{2}$$

Corollary.

$$r = (s - a) \tan (A/2), \tag{3a}$$

$$\begin{aligned} \text{for } \tan (A/2) &= \sqrt{\left\{\frac{(s-b)\ (s-c)}{s\,(s-a)}\right\}} \\ &= \sqrt{\ }\{s\,(s-a)\ (s-b)\ (s-c)\}\ /\ \{s\,(s-a)\}\,, \\ &= r/(s-a). \end{aligned}$$

Similarly,

$$r = (s - b) \tan (B/2), \qquad (3b)$$
$$\text{and } r = (s - c) \tan (C/2). \qquad (3c)$$

(3) *Ex-circle :* Again if AI_1 (Fig. 4.6) is the internal bisector of the angle A, and BI_1, CI_1 are the external bisectors of angles B and C, then I_1 is the centre of the ex-circle opposite A. Let I_1X_1, I_1Y_1, I_1Z_1 be perpendiculars to BC, CA, AB respectively; each of the perpendiculars has length r_1.

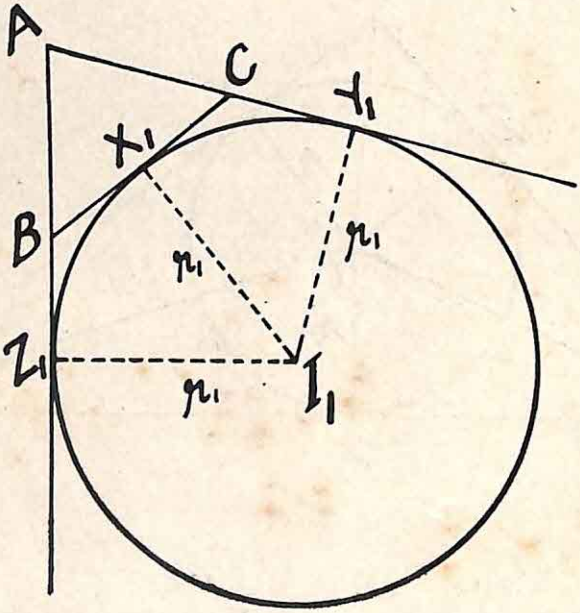


Fig. 4.6

$$\begin{aligned}\text{Then } \triangle &= \triangle I_1CA + \triangle I_1AB - \triangle I_1CB,\\ &= (1/2)\, CA.\, I_1Y_1 + (1/2)\, AB.\, I_1Z_1 - (1/2)\, BC.\, I_1X_1,\\ &= (1/2)\, r_1 (b + c - a) = r_1 (s - a).\end{aligned}$$

$$\text{Hence} \quad \boxed{r_1 = \frac{\triangle}{s - a}.} \qquad (3a)$$

Example 6. Show that $r = 4R \sin (A/2) \sin (B/2) \sin (C/2)$.

Solution. $$4R\ \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = 4.\frac{abc}{4\triangle}\cdot\sqrt{\left\{\frac{(s-b)\ (s-c)}{bc}\right\}}\cdot$$
$$\sqrt{\left\{\frac{(s-c)\ (s-a)}{ca}\right\}}\cdot\sqrt{\left\{\frac{(s-a)\ (s-b)}{ab}\right\}},$$
$$= \frac{(s-a)\ (s-b)\ (s-c)}{\triangle},$$
$$= \frac{s\ (s-a)\ (s-b)\ (s-c)}{s\triangle},$$
$$= \frac{\triangle^2}{s\triangle} = \frac{\triangle}{s} = r.$$

Example 7. Prove that $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{1}{r}$.

Solution. $$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3} = \frac{s-a}{\triangle}+\frac{s-b}{\triangle}+\frac{s-c}{\triangle},$$
$$= \{(s-a)+(s-b)+(s-c)\}\ \triangle,$$
$$= s/\triangle = 1/r.$$

Example 8. Prove that $r_1 + r_2 + r_3 = 4R + r$.

Solution. The given problem is equivalent to showing that $r_1+r_2+r_3-r=4R$.

$$r_1+r_2+r_3-r = \frac{\triangle}{s-a}+\frac{\triangle}{s-b}+\frac{\triangle}{s-c}-\frac{\triangle}{s},$$
$$= \triangle\left\{\frac{(s-b)+(s-a)}{(s-a)\ (s-b)}\right\} + s\triangle\left\{\frac{s-(s-c)}{(s-c)\ s}\right\},$$
$$= \frac{c\triangle}{(s-a)\ (s-b)}+\frac{c\triangle}{s(s-c)},$$
$$= \frac{c\triangle\{s(s-c)+(s-a)\ (s-b)\}}{s(s-a)\ (s-b)\ (s-c)},$$
$$= c\triangle\{2s^2-s(a+b+c)+a\,b\}/\triangle^2,$$
$$= \frac{abc}{\triangle} = 4.\ \frac{abc}{4\triangle} = 4R.$$
$$\therefore\qquad r_1+r_2+r_3 = 4R+r.$$

Example 9. Prove that
$$(r_1-r)\ (r_2-r)\ (r_3-r) = 4R\ r^2.$$

Solution.

$$\begin{aligned}(r_1-r)\ (r_2-r)\ (r_3-r) &= \left(\frac{\triangle}{s-a}-\frac{\triangle}{s}\right)\left(\frac{\triangle}{s-b}-\frac{\triangle}{s}\right)\left(\frac{\triangle}{s-c}-\frac{\triangle}{s}\right),\\ &= \triangle^3.\ \frac{a}{s(s-a)}\cdot\ \frac{b}{s(s-b)}\cdot\ \frac{c}{s(s-c)},\\ &= \frac{\triangle^3\ abc}{s^2\{s(s-a)\ (s-b)\ (s-c)\}},\\ &= \frac{\triangle^3\ abc}{s^2\ \triangle^2},\\ &= 4.\ \frac{abc}{4\triangle}\cdot\left(\frac{\triangle}{s}\right)^2,\\ &= 4R\ r^2.\end{aligned}$$

EXERCISES

In any triangle ABC, prove that

1. $\triangle = 2R^2 \sin A \sin B \sin C$.
2. $a \cos A + b \cos B + c \cos C = 4R \sin A \sin B \sin C$
3. $\sin A + \sin B + \sin C = \frac{s}{R}$.
4. $r = a \sin \frac{B}{2} \sin \frac{C}{2} \sec \frac{A}{2}$.
5. $R\ r (\sin A + \sin B + \sin C) = \triangle$.
6. $\cos A + \cos B + \cos C = 1 + \frac{r}{R}$.
7. $r_1 = s \tan (A/2)$.
8. $r_1 = (s - c) \cot (B/2) = (s - c) \cot (C/2)$
9. $r_1 = 4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$.
10. $\frac{b-c}{r_1} + \frac{c-a}{r_2} + \frac{a-b}{r_3} = 0$.
11. $r_1 r_2 + r_2 r_3 + r_3 r_1 = s^2$.
12. $a = (r_2 + r_3) \sqrt{\left\{\frac{r\, r_1}{r_2 r_3}\right\}}$.
13. $r_1 + r_2 = c \cot \frac{C}{2}$.
14. $r\, r_1 = r_2 r_3 \tan^2 \frac{A}{2}$.

15. $16R^2 r r_1 r_2 r_3 = a^2 b^2 c^2$.

16. $a (rr_1 + r_2r_3) = b (r r_2 + r_3r_1) = c (rr_3 + r_1 r_2)$.

17. $r_1 r_2 r_3 = r^3 \cot^2 \frac{A}{2} \cot^2 \frac{B}{2} \cot^2 \frac{C}{2}$.

18. $(1/r)^2 + (1/r_1)^2 + (1/r_2)^2 + (1/r_3)^2 = (a^2+b^2+c^2)/\triangle^2$.

19. $(r_3 - r) \cot \frac{C}{2} = c$.

20. Show that in an equilateral triangle
$r_1 = r_2 = r_3 = 3r = (R/2)$.

4.5. Regular Polygons

Let O be the centre of the circles circumscribed about and inscribed in a regular polygon of n sides (Fig. 4.7). Let R and r be the radii of the former and the latter circles respectively, and let *a* be the length of a side of the polygon.

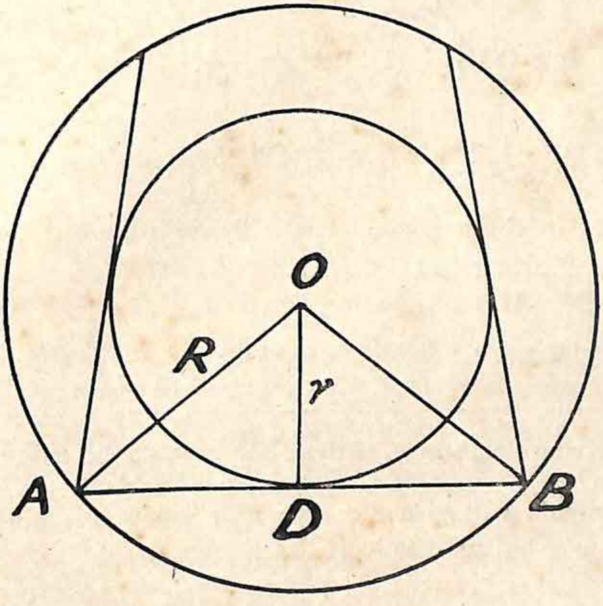


Fig. 4.7

If AB be a side of the polygon, and D is the point where the inscribed circle touches it, the angle AOB is $2\pi/n$ and the angle AOD is π/n. Thus

$$a = 2AD = 2OA. \text{ Sin } \angle \text{ AOD},$$

$$= 2R \sin \frac{\pi}{n}.$$

Therefore

$$\boxed{R = \frac{a}{2 \sin \frac{\pi}{n}}.} \tag{1}$$

Also,

$$a = 2 \text{ OD}. \tan \angle \text{ AOD},$$

$$= 2r \tan \frac{\pi}{n}.$$

Therefore

$$\boxed{r = \frac{a}{2 \tan \frac{\pi}{n}}.} \tag{2}$$

The area of the triangle OAB $= (1/2)\ R^2 \sin \frac{2\pi}{n}$.

Also this $= (1/2)\ a.r = r^2 \tan \frac{\pi}{n} = \frac{a^2}{4} \cot \frac{\pi}{n}$.

Corollary. The area of the polygon is given by the following formulae :

$$\boxed{\begin{array}{c}\text{Area of the polygon}\\ = (\ 1/2\ )\ n\ R^2 \sin\frac{2\pi}{n} = n\, r^2 \tan \frac{\pi}{n} = \frac{na^2}{4} \cot \frac{\pi}{n}\end{array}} \tag{3}$$

for the area under consideration is n times the area of the triangle OAB.

Note. This formula will give the area of a circle of radius R by using the concept of a limit which you will learn later.

Example. The length of each side of a regular hexagon is 10 centimetres. Find (i) the radius of its inscribed circle, (ii) the radius of its circumscribing circle, and (iii) its area, giving the answers correct to two decimal places.

Solution. The angle subtended by a side of a regular hexagon at the centre of the polygon $= \frac{360°}{6} = 60°$. Therefore, we have $5 = r \tan 30° = R \sin 30°$.

$$\therefore r = 5 \cot 30°,$$
$$= 5\sqrt{3} = 5 \times 1{\cdot}732 = 8{\cdot}66 \text{ cm. (approx.)}$$
$$\text{Also } R = 5 \operatorname{cosec} 30° = 10 \text{ cm.}$$

Again, the area $= 6 \times r \times 5$ sq. cm,
$= 30 \times \sqrt{3} = 51{\cdot}96$ sq. cm.

EXERCISES

1. Find the area of (i) a hexagon (ii) an octagon and (iii) a decagon, each being a regular figure of side 5 cm.

2. One side of a regular decagon is 10 cm. Find the radii of the inscribed and circumscribed circles.

3. If an equilateral triangle and a regular hexagon have the same perimeter, prove that their areas are as 2 : 3.

4. If a regular pentagon and a regular decagon have the same perimeter, prove that their areas are as $2 : \sqrt{5}$.

5. Find the difference between the areas of a regular octagon and a regular hexagon if the perimeter of each is 12 decimetres.

6. Show that the areas of a regular hexagon and a regular octagon inscribed in a given circle are as $\sqrt{27} : \sqrt{32}$.

7. Given that the area of a regular polygen of n sides circumscribed about a circle is to the area of the circumscribed polygon of 2n sides as 3:2, find n·

8. Two regular polygons of n sides each are inscribed in and circumscribed about a given circle. If their areas are as 3 : 4, find n.

9. The sides of a triangle are respectively a side of a regular pentagon, hexagon and decagon inscribed in a circle. Prove that the triangle is right-angled.

10. If a be the side of a regular polygon of n sides, R and r the radii of the circumscribed and inscribed circles, prove that

$$R + r = \frac{a}{2} \cot \frac{\pi}{2n}.$$

ASSORTED EXERCISES ON CHAPTER 4

1. In any triangle ABC, prove that

(i) $a^2 \sin 2B + b^2 \sin 2A = 2ab \sin C$.

(ii) $c \sin\left(\frac{C}{2} + A\right) = (c + a) \sin \frac{C}{2}$.

(iii) $(b + c) \cos A + (c + a) \cos B + (a + b) \cos C = a + b + c$.

(iv) $a (\cos C - \cos B) = 2 (b - c) \cos^2 \frac{A}{2}$.

(v) $\frac{\sin (A - B)}{\sin (A - B)} = \frac{a^2 - b^2}{c^2}$.

(vi) $\frac{b^2 - c^2}{a^2} \sin 2A + \frac{c^2 - a^2}{b^2} \sin 2B + \frac{a^2 - b^2}{c^2} \sin 2C = 0$.

(vii) $a^2 (\cos^2 B - \cos^2 C) + b^2 (\cos^2 C - \cos^2 A) + c^2 (\cos^2 A - \cos^2 B) = 0$.

(viii) $a^3 \sin (B - C) + b^3 \sin (C - A) + c^3 \sin (A - B) = 0$.

(ix) $\frac{a \sin (B - C)}{b^2 - c^2} + \frac{b \sin (C - A)}{c^2 - a^2} = \frac{c \sin (A - B)}{a^2 - b^2}$.

(x) $\frac{a^2 \sin (B - C)}{\sin B + \sin C} + \frac{b^2 \sin (C - A)}{\sin C + \sin A} + \frac{c^2 \sin (A - B)}{\sin A + \sin B} = 0$.

2. If a, b, c are in H. P., then $\sin^2 (A/2)$, $\sin^2 (B/2)$, $\sin^2 (C/2)$ are also in H.P.

3. If $a \cos^2 (C/2) + c \cos^2 (A/2) = \frac{3b}{2}$, then a, b, c are in A.P.

4. If a^2, b^2, c^2 are in A.P., then cot A, cot B, cot C are also in A.P.

5. $R = \frac{abc (\cot A + \cot B + \cot C)}{a^2 + b^2 + c^2}$.

6. $\text{Cos } A = \frac{2R + r - r_1}{2R}$.

7. $\frac{r_1}{bc} + \frac{r_2}{ca} + \frac{r_3}{ab} = \frac{1}{r} - \frac{1}{2R}$.

8. $\frac{r}{R} = \frac{a \cos A + b \cos B + c \cos C}{a + b + c}$.

9. The area of the inscribed circle is to the area of the triangle as

$$\pi : \cot (A/2) \cot (B/2) \cot (C/2).$$

10. If a, b, c are in A.P., then r_1, r_2, r_3 are in H.P.

11. If $r_1 = r_2 + r_3 + r$, then the triangle is right-angled.

12. If A, A_1, A_2, A_3 be the areas of the in-circle and the ex-circles, then

$$\frac{1}{\sqrt{A}} = \frac{1}{\sqrt{A_1}} + \frac{1}{\sqrt{A_2}} + \frac{1}{\sqrt{A_3}}.$$

13. Of two regular polygons of n sides, one is circumscribed about and the other is inscribed in a given circle. Prove that the perimeters of the circumscribed and inscribed polygons and the circle are as

$$\sec(\pi/n) : 1 : \pi/n \csc(\pi/n),$$

and that the areas of the polygons are as $1 : \cos^2 \frac{\pi}{n.}$.

14. Prove that the area of a regular polygon of 2n sides circumscribed about a circle is the harmonic mean between the areas of the inscribed regular polygon of 2n sides and circumscribed regular polygon of n sides.

15. If R, r be the radii of the circumscribed and inscribed circles of a regular polygon of the same area but double the number of sides, show that

$$R'^2 = R\,r, \text{ and } r'^2 : r^2 = 1/2\,(R + r).$$

16. If A_1 be the area of a regular polygon of 2n sides inscribed in a circle, and A_2, A_3 be the areas of regular polygons of n sides inscribed in and circumscribed about the same circle, prove that A_1, A_2, A_3 are in G.P.

Chapter 5

Logarithms

5.1. Introduction

Whenever we have a problem involving numerical computations, logarithms prove to be an indispensable tool. They are so useful that some people are inclined to call them 'the servants of man'. In the present chapter we propose to tell you what logarithms are, and how we can use them. In the next chapter we shall tell you about applications of logarithms to solution of triangles.

For defining logarithms we shall need certain properties of real numbers which we shall state without proof :

I. For every positive real number a, not equal to unity, and every positive real number N, there is one and only one real number x satisfying the equation

$$a^x = N.$$

II. For any given real number $a > 1$ and for any real number $N > 1$, the solution of the equation $a^x = N$ must be positive.

III. $1^x = 1, \forall x \in \mathbf{R}$.

IV. $a^0 = 1, \forall a \in \mathbf{R}$.

V. $(a^x)^y = a^{xy}, \forall$ a, x, y in $\mathbf{R}$.

VI. $a^x . b^x = (ab)^x, \forall$ a, b, x in $\mathbf{R}$.

Definition. If a be a positive real number, not equal to unity, and N be a positive real number, then the real number x which satisfies the equation $a^x = N$, is called the *logarithm* of N to the *base* a. We write the logarithm of N to the base a as $\log_a N$. Thus

$$a^x = N \iff x = \log_a N.$$

Of the two equivalent statements written above, the first is said to be in exponential form and the second is said to be in logarithmic form. Whenever we write $\log_a N$, it is always understood that N is a positive real number and that a is a positive real number, different from unity.

In any particular discussion if it is understood without any ambiguity as to what base is being used, then we may omit the suffix indicating the base.

Examples 1. Since $2^4 = 16$, therefore $\log_2 16 = 4$.

2. Since $(1/2)^{-3} = 8$, therefore $\log_{1/2} 8 = -3$.
3. Since $7^1 = 7$, therefore $\log_7 7 = 1$.
4. Sinee $(\cdot 1)^{-4} = 10000$, therefore $\log_{\cdot 1} 10000 = -4$.
5. Since $(10)^3 = 1000$, therefore $\log_{10} 1000 = 3$.

Example 1. Solve the following equations :

(i) $\log_3 x = 4$; (ii) $\log_a 64 = 3$; (iii) $\log_{1/2} 16 = x$.

Solution : (i) $\log_3 x = 4 \iff 3^4 = x \iff x = 81$.

(ii) $\log_a 64 = 3 \iff a^3 = 64 \iff a^3 = 4^3 \iff a = 4$.

(iii) $\log_{1/2} 16 = x \iff (1/2)^x = 16 \iff (1/2)^x = (1/2)^{-4} \iff x = -4$.

EXERCISES

1. Give the logarithmic form of each of the following statements :

(i) $2^6 = 64$. (ii) $3^4 = 81$. (iii) $7^{-2} = \frac{1}{49}$.

(iv) $6^0 = 1$. (v) $27^{2/3} = 9$. (vi) $(32)^{2/5} = 4$.

2. Find x, y or a in each of the following equations :

(i) $\log_2 x = 4$. (ii) $\log_3 x = -1$. (iii) $\log_4 x = 0$.

(iv) $\log_9 x = 3/2$. (v) $\log_7 x = -2$. (vi) $\log_{1/2} x = 3$.

(vii) $\log_a 16 = 4$. (viii) $\log_a 27 = 3$. (ix) $\log_a 1000 = 3$.

(x) $\log_a \frac{1}{32} = -5$. (xi) $\log_a 1 = 0$. (xii) $\log_a \frac{1}{7} = -1$.

(xiii) $\log_5 625 = y$. (xiv) $\log_2 32 = y$. (xv) $\log_{\frac{3}{2}} \frac{27}{8} = y$.

(xvi) $\log_{\frac{1}{2}} 16 = y$. (xvii) $\log_{64} 4 = y$. (xviii) $\log_{216} 6 = y$.

5.2. Properties of Logarithms

We shall now prove some important properties of logarithms.

Theorem 1. For any two positive real numbers M and N,

$$\boxed{\log_a (MN) = \log_a M + \log_a N.}$$

Proof : Let $\log_a M = x$ and $\log_a N = y$, so that

$$a^x = M \text{ and } a^y = N.$$

Now $MN = a^x . a^y = a^{x+y}$,
whence by definition,

$$\log_a (MN) = x + y = \log_a M + \log_a N.$$

Remark. The above theorem can be immediately extended to the product of any finite number of positive real factors. That is, we can prove that *if* $M_1, M_2, \ldots, M_k$ be any positive real numbers, then

$$\log_a (M_1 M_2 \ldots\ldots M_k) = \log_a M_1 + \log_a M_2 + \ldots\ldots + \log_a M_k.$$

Example : $\log_{10} 42 = \log_{10} (2.3.7)$,
$= \log_{10} 2 + \log_{10} 3 + \log_{10} 7.$

Theorem 2. For any two positive real nnumbers M and N,

$$\boxed{\log_a (M/N) = \log_a M - \log_a N.}$$

Proof : Let $\log_a M = x$ and $\log_a N = y$, so that

$$a^x = M \text{ and } a^y = N.$$

Now $M/N = a^x/a^y = a^{x-y}$,
whence by definition,

$$\log_a (M/N) = x - y = \log_a M - \log_a N.$$

Example : $\log_{10} (22/7) = \log_{10} 22 - \log_{10} 7$,
$= \log_{10} (11.2) - \log_{10} 7$,
$= \log_{10} 11 + \log_{10} 2 - \log_{10} 7.$

Theorem 3. For any positive real number M and any positive real number p,

$$\boxed{\log_a M^p = p \log_a M.}$$

Proof : Let $\log_a M = x$, so that $a^x = M$.
Now $M^p = (a^x)^p = a^{xp} = a^{px}$,
whence, by definition,

$$\log_a M^p = p\,x = p \log_a M.$$

Examples : 1. $\log_{10} 216 \equiv \log_{10} 6^3 \equiv 3 \log_{10} 6.$

2. $\log_4 (27)^{\frac{1}{2}} = \log_4 (3^3)^{\frac{1}{2}} = \log_4 3^{\frac{3}{2}} = \frac{3}{2} \log_4 3.$

Example 2. Express $\log_{10}\sqrt{\{a^5/(c^4\ b^4)\}}$
as a sum of logarithms,

Solution.
$$\begin{aligned}\log_{10}\{\sqrt{(a^5/(c^3b^4))}\} &= (1/2)\log_{10}\{a^5/(c^3b^4)\},\\ &= (1/2)[\log_{10}a^5 - \log_{10}(c^3b^4)],\\ &= (1/2)[5\log_{10}a - (\log_{10}c^3 + \log_{10}b^4)],\\ &= (1/2)[5\log_{10}a - (3\log_{10}c + 4\log_{10}b)],\\ &= (5/2)\log_{10}a - 3/2\log_{10}c - 2\log_{10}b.\end{aligned}$$

Example 3. Show that the following is an identity :

$$\log_a 48 - \log_a 24 + \log_a 35 - \log_a 10 = \log_a 7.$$

Solution.
$$\begin{aligned}\text{L. H. S.} &= (\log_a 48 - \log_a 24) + (\log_a 35 - \log_a 10),\\ &= \log_a \frac{48}{24} + \log_a \frac{35}{10},\\ &= \log_a 2 + \log_a \tfrac{7}{2},\\ &= \log_a (2 \times \tfrac{7}{2}),\\ &= \log_a 7.\end{aligned}$$

Example 4. Solve the equation

$$\log_3 4 + \log_3(2x - 1) = \log_3(x + 2) - 1.$$

Solution. The given equation can be written as

$$\log_3 4 + \log_3(2x - 1) - \log_3(x + 2) = -1,$$

or as $\log_3\{4(2x-1)\} - \log_3(x+2) = -1,$

or as $\log_3\left[\dfrac{4(2x-1)}{x+2}\right] = -1.$

The above equation can be written in exponential form as

$$3^{-1} = \frac{4(2x-1)}{x+2},$$

or as $x + 2 = 12(2x - 1),$

which gives on solving, $x = 14/23$.

On substituting $x = 14/23$ in the given equation we find that it satisfies the given equation.

Hence the solution-set is $\{14/23\}$.

EXERCISES

1. Express each of the following logarithms as an algebraic sum of logarithms. The bases to which the logarithms are to be considered have not been indicated. Use any positive number (other than 1) of your choice as base :

(i) $\log(231 \times 476)$. (ii) $\log(2^{10} . 7^4 . 9^{-3})$.

(iii) $\log\left[\dfrac{x^2 y^3 z^4}{p\,q\,r}\right]$. (iv) $\log\left[\sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}\right]$.

2. Express each of the following in logarithmic form. (Take an arbitrary positive number *a*, different from 1, as the base):

(i) $s = (1/2)\, g\, t^2$. (ii) $V = (4/3)\, \pi r^3$.

(iii) $t = 2\pi \sqrt{\dfrac{l}{g}}$. (iv) $F = \dfrac{k\, m_1\, m_2}{r^2}$.

3. Given $\log_{10} 2 = {\cdot}3010$, $\log_{10} 3 = {\cdot}4771$, find the value of each of the following:

(i) $\log_{10} 2^6$. (ii) $\log_{10} 3^{-11}$.

(iii) $\log_{10} 24$. (iv) $\log_{10} \dfrac{81}{32}$.

(v) $\log_{10} \sqrt{1536}$. (vi) $\log_{10} \sqrt{\dfrac{1024}{243}}$.

4. Use the laws of logarithms to show that each of the following equations is an identity:

(a is a positive number different from 1.)

(i) $\log_a 48 - \log_a 168 + \log_a 35 = \log_a 10$.

(ii) $3 \log_a x - 2 \log_a y + 4 \log_a \sqrt{x} - (1/2) \log y = (1/2) \log_a \left\{ \dfrac{x^{10}}{y^5} \right\}$.

5. Solve each of the following equations:

(i) $\log_3 54 - (1/2) \log_3 4 = x$. (ii) $\log_4 x + 3 \log_4 x = \log_4 81$.

(iii) $\log_4 64 - \frac{1}{2}\log_4 256 = 2x+1$. (iv) $\frac{1}{7}\log_4 x^2 + \frac{5}{7} \log_4 x = 2$.

(v) $\log_2 28 - \log_2 7x = 2$. (vi) $\log_3 3 + \log_3 (x+1) - \log_3 (2x-7) = 4$.

(vii) $\log_9 | 2x + 1 | - \log_9 | x+2 | = \frac{1}{2}$.

6. Prove :

(i) $\dfrac{\log_a (x + h) - \log_a x}{h} = \log_a \left(1 + \dfrac{h}{x} \right)^{\frac{1}{h}}$.

(ii) $\log_a \left\{ \dfrac{x}{\sqrt{x^2 + b^2} - b} \right\} = \log_a \left\{ \dfrac{\sqrt{x^2 + b^2} + b}{x} \right\}$.

(iii) $\log_a \dfrac{2x - \sqrt{(4x^2 - 1)}}{2x + \sqrt{(4x^2 - 1)}} = 2 \log_a \{2x - \sqrt{(4x^2 - 1)}\}$.

7. Prove that the equation

$$\log_a (1/N) = - \log_a N$$

is true for every positive real number N, and any positive base *a* different from 1.

8. Prove that for any two positive bases *a* and *b* different from 1,

$$\log_b N = \frac{\log_a N}{\log_a b},$$

9. Prove that the equation $\log_a b . \log_b a = 1$ is true for every pair of positive numbers, a, b, different from 1.

10. Prove that if $a > 1$, and $M > N > 0$, then $\log_a M > \log_a N$.

11. Prove that if a and b be two positive numbers, both greater than 1, then $\log_a b > 0$.

5.3. Common Logarithms

Two systems of logarithms are generally used in mathematics. One of these, called *natural logarithms,* is due to Napier. It has, as its base, the irrational number e (approximately equal to 2·71828). It is generally used in advanced mathematics. The other one, called common logarithms, is due to Briggs. It has, as its base, the number 10. This system is very convenient for numerical computations, especially because of the fact that 10 also happens to be the base of our decimal system of notation. We shall throughout use the system of common logarithms and shall therefore not write the base. Whenever no base is written, it would be understood that the base is 10. Thus, we shall write simply log 2, log 3, ... instead of $\log_{10} 2$, $\log_{10} 3$,......; also log N would mean $\log_{10} N$.

The English mathematician Briggs computed and published in 1624, a table of common logarithms of all integers from 1 to 20000, and from 90,000 to 100,000. Since the logarithms of most integers happen to be irrational numbers, the entries in the table were only rational approximations. (Of course, they had a very high degree of accuracy. In fact, he had computed the logarithms correct to 14 decimal places.) For ordinary purposes it is enough to use a much smaller table than that of Briggs, and also to have a much lower degree of accuracy than he had. We shall use a *four-figure table* (i.e., a table in which the logarithms are given correct to four places of decimals), but before we do so, we shall learn the use of scientific notation and a few more facts about logarithms.

5.4. Scientific Notation

Consider some numbers, say, 29·2, 629·4, ·0437 and 1223. We may write these as

$$29{\cdot}2 = 2{\cdot}92 \times 10, \qquad {\cdot}0437 = 4{\cdot}37 \times 10^{-2},$$

$$629{\cdot}4 = 6{\cdot}294 \times 10^{2}, \qquad 1223 = 1{\cdot}223 \times 10^{3}.$$

The above manner of expressing the numbers 29·2, 629·4, ·0437 and 1223 shows that each of them can be expressed as the product of a number lying between 1 and 10, and an integral power of 10. We can do this for any positive number whatever. That is, *if N be any positive number, then it is possible to find a number n (where $1 \leq n < 10$) and an integer r such that*

$$N = n \times 10^r.$$

The above notation is called *scientific notation.*

Examples. (i) The distance that light travels in one year is called a *light-year,* which is approximately equal to 5,870,000,000,000 mi. In scientific notation this is equal to $5{\cdot}87 \times 10^{12}$ mi.

(ii) The thickness of an oil film is about ·0,000,005 cm. In scientific notation this is 5×10^{-7} cm.

(iii) The mass of a molecule of water is about ·00,000,000,000,000,000,000,003 gm. In scientific notation, this is 3×10^{-23} gm.

As we shall see later, it is convenient to write a number in scientific notation before finding its logarithm.

5.5. Logarithm of a Number lying between 1 and 10

Theorem 5. If $1 \leqslant N < 10$, then $0 \leqslant \log N < 1$.

Proof. Let N be any number such that $1 \leqslant N < 10$. Since the function *log* to any base exceeding 1 is a strictly increasing function (See ex. 10, page 123), therefore

$$\log_{10} 1 \leqslant \log_{10} N < \log_{10} 10. \qquad \ldots\ldots \text{ (i)}$$

Now, logarithm of 1 to any base (other than unity) being zero, we have $\log_{10} 1 = 0$. Also, since the logarithm of the base itself is unity, therefore $\log_{10} 10 = 1$.

We may, therefore, write (i) as

$$0 \leqslant \log_{10} N < 1,$$

$$\text{i.e., } 0 \leqslant \log N < 1,$$

in view of our convention regarding omitting the base 10.

5.6. Characteristic and Mantissa

Consider any positive number N. Expressing N in scientific notation, we have

$$N = n \times 10^{r}, \qquad \ldots\ldots \text{ (i)}$$

where $1 \leqslant n < 10$, and r is an integer.

By taking logarithms of both sides of (i) to the base 10, we have

$$\begin{aligned} \log N &= \log n + \log 10^{r}, \\ &= \log n + r \log 10, \\ &= \log n + r. \end{aligned} \qquad \ldots\ldots \text{ (ii)}$$

Since, by theorem 4, $0 \leq \log n < 1$, therefore we find from (ii) that *the common logarithm of any positive number can be expressed as the sum of an integer r and a number between 0 and 1. The integer r is called the characteristic and the part between 0 and 1 is called the mantissa.* Thus, for example,

(i) if $\log N = 2{\cdot}1076$, then
characteristic is 2 and mantissa is ·1076.

(ii) If $\log N = 0{\cdot}3010$, then
characteristic is zero and mantissa is $\cdot3010$.

(iii) If $\log N = -4 + \cdot4771$, then characteristic is -4, and mantissa is $\cdot4771$. To determine the logarithm of a number, we have to find both the characteristic and the mantissa. The following theorem gives us a rule for finding out the characteristic of the logarithm of a number.

Theorem. If the decimal point in a number is m places to the right of the standard position, then the characteristic of the logarithm is m ; and if the decimal point is m places to the left of the standard position, then the characteristic of the logarithm is $-m$.

Proof. Let N be any positive number. If the decimal point in N is m places to the right of the standard position, then the expression for N in scientific notation is of the form

$$N = n \times 10^{m}, \qquad (1 \leqslant n < 10)$$

and therefore the characteristic of log N is m.

If, on the other hand, the decimal point in N be m places to the left of the standard position, then the expression for N in scientific notation is of the form

$$N = n \times 10^{-m}, \qquad (1 \leqslant n < 10)$$

and therefore the characteristic of log N is $-m$.

Remark. If the decimal point in N is in standard position, then the characteristic of log N is zero.

From the above theorem and the remark following it we find that

(i) if $N \geqslant 10$, and there are k digits to the left of the decimal point, then the decimal point is $k - 1$ places to the right of the standard position, so that characteristic of $\log N = k - 1$.

(ii) If $0 < N < 1$, and there are k zeros between the decimal point and the first non-zero digit in N, then the decimal point is $k + 1$ places to the left of the standard position, so that the characteristic of $\log N = -(k + 1)$.

(iii) If $\leqslant N < 10$, then characteristic of log N is zero.

From the above discussion, we have the following rule for determining the characteristic of the logarithm of a number :

Rule. If $N \geqslant 10$, the characteristic is positive, and is one less than the number of digits to the left of the decimal point.

If $N < 1$, the characteristic is negative, and is numerically equal to one more than the number of zeros between the decimal point and the first non-zero digit in N.

If $1 \leqslant N < 10$, the characteristic is zero.

Remark. It is, however, advisible to express a number in scientific notation to find its characteristic.

Example 5. Express each of the following numbers in scientific notation, and hence find the characteristics of their logarithms :

4·718, 47·18, 471·8, ·04718, ·4718.

Solution :

N	Expression for N in scientific notation	Characteristic of log N
4·718	$4·718 \times 10^0$	0
47·18	$4·718 \times 10^1$	1
471·8	$4·718 \times 10^2$	2
·04718	$4·718 \times 10^{-2}$	-2
·4718	$4·718 \times 10^{-1}$	-1.

Example 6. Find by inspection the characteristic of the logarithm of each of the following numbers :

2,634, ·00,635, 16·74, 1·245, ·4718.

Solution. Writing 2,634 as 2634·0 we find that it has 4 digits to the left of the decimal point, so that the characteristic acteristic of log ·00635 is -3.

Since $·00635 < 1$ and since there are 2 zeros between the decimal point and the first non-zero digit, therefore the characteristic of log ·00635 is -3.

Since there are 2 digits to the left of the decimal point in 16·74, therefore the characteristic of log 16·74 is 1.

Since $1 < 1·245 < 10$, therefore the characteristic of log 1·245 is 0.

Since $·4718 < 1$, and since there is no zero between the decimal point and first non-zero digit in ·4718, therefore the characteristic of log ·4718 is -1.

Example 7. Given $\log 2 = ·3010$, find the number of digits in 2^{31}.

Solution : Let $N = 2^{31}$.

Taking logarithms of both sides, we have

$$\begin{aligned} \log N &= 31 \log 2, \\ &= 31 \times ·3010, \\ &= 9·3310. \end{aligned}$$

$\therefore$ Characteristic of $\log N = 9$, so that the number of digits to the left of the decimal point in N is 10. But N is an integer. Therefore the number of digits in N is 10.

Example 8. Given $\log 3 = \cdot 4771$, find the position of the first significant digit in 3^{-29}.

Solution : Let $N = 3^{-29}$.

Taking logarithms of both sides, we have

$$\begin{aligned} \log N &= -29 \log 3, \\ &= -29 \times \cdot 4771, \\ &= -13{\cdot}8359, \\ &= -14 + \cdot 1641, \end{aligned}$$

so that the characteristic of $\log N$ is -14.
Since the characteristic is negative, therefore $N < 1$. Also, there must be 13 zeros between the decimal point and the first non-zero digit in N, i.e. the first significant digit is the fourteenth digit to the right of the decimal point.
The following theorem gives an important result regarding mantissas :

Theorem 5. The mantissa of the logarithm of a number N is determined uniquely by the succession of digits in N, and is the same for all numbers having the same succession of digits.

Proof : Let the expression for N in scientific notation be of the form

$$n \times 10^r, \text{ where } 1 \leqslant n < 10, \text{ and r is an integer.}$$

Then the mantissa of $\log N = \log n$.

If M be *any* number having the same succession of digits as N, then M differs from N by an integral power of 10, that is,

$$\begin{aligned} & M = n \times 10^k, \text{ for some integer k,} \\ \text{or} \quad & M = n \times 10^{k+r}, \text{ where } 1 \leqslant n < 10, \\ & \text{and } k + r \text{ is an integer.} \end{aligned}$$

This shows that the mantissa of $\log M$ is also equal to $\log n$.

Thus $\log M$ and $\log N$ have the same mantissa.

Corollary. If the expression for a number N in scientific notation be $n \times 10^r$, where $1 \leqslant n < 10$ and r is an integer, then the mantissa of $\log N$ is $\log n$.

The above result shows that if the logarithms of all numbers between 1 and 10 be known, then we can write down the mantissa of the logarithm of any number.

Example 9. Given $\log 31{\cdot}15 = 1{\cdot}4935$, find the value of (i) $\log 3115$, (ii) $\log 31150$, (iii) $\log \cdot 3115$, (iv) $\log \cdot 003115$.

Solution. (i) Characteristic of $\log 3115 = 3$.

$$\begin{aligned} \text{mantissa of } \log 3115 &= \text{mantissa of } \log 31{\cdot}15, \\ &= \cdot 4935. \end{aligned}$$

$$\therefore \log 3115 = 3{\cdot}4935.$$

(ii) Characteristic of $\log 31150 = 4$;
mantissa of $\log 31150 =$ mantissa of $\log 31{\cdot}15$,
$= {\cdot}4935$.
$\therefore \log 31150 = 4{\cdot}4935$.

(iii) Characteristic of $\log {\cdot}3115 = -1$;
mantissa of $\log {\cdot}3115 =$ mantissa of $\log 31{\cdot}15$,
$= {\cdot}4935$.
$\therefore \log {\cdot}3115 = -1 + {\cdot}4935$
$= 9{\cdot}4935 - 10$.

(iv) Characteristic of $\log {\cdot}003115 = -3$;
mantissa of $\log {\cdot}003115 =$ mantissa of $\log 31{\cdot}15$,
$= {\cdot}4935$.
$\therefore \log {\cdot}003115 = -3 + {\cdot}4935$
$= 7{\cdot}4935 - 10$.

Aliter: $31{\cdot}15 = 3{\cdot}115 \times 10$, so that
$\log 31{\cdot}15 = \log 3{\cdot}115 + 1$,
which gives $\log 3{\cdot}115 = {\cdot}4935$.
Now $3115 = 3{\cdot}115 \times 10^3$,
so that $\log 3115 = \log 3{\cdot}115 + 3 = 3{\cdot}4935$.
$31150 = 3{\cdot}115 \times 10^4$,
so that $\log 31150 = \log 3{\cdot}115 + 4 = 4{\cdot}4935$.
${\cdot}3115 = 3{\cdot}115 \times 10^{-1}$,
so that $\log {\cdot}3115 = \log 3{\cdot}115 - 1 = {\cdot}4935 - 1 = 9{\cdot}4935 - 10$.
${\cdot}003115 = 3{\cdot}115 \times 10^{-3}$,
so that $\log {\cdot}003115 = \log 3{\cdot}115 - 3 = {\cdot}4935 - 3 = 7{\cdot}4935 - 10$.

Remark. It is usual to express a negative characteristic —1 as 9—10, —2 as 8—10, —3 as 7—10, etc.

Example 10. Given $\log 670{\cdot}5 = 2{\cdot}8264$, find N, if
(i) $\log N = 1{\cdot}8264$, (ii) $\log N = 8{\cdot}8264 - 10$.

Solution. (i) Since log N and log 670·5 have the same mantissa, therefore N and 670·5 have the same succession of digits. Also, since the characteristic of log N is 1, therefore the decimal point in N must be to the right of the standard position by one place.

Therefore $\log N = \log 67{\cdot}05$, i.e. $N = 67{\cdot}05$.

(ii) Since log N and log 670·5 have the same mantissa, therefore N and 670·5 have the same succession of digits. Also, since the characteristic of log N is 8—10, i.e. —2, therefore the decimal point in N is 2 places to the left of the standard position.

Therefore log N = log ·06705, i.e. N = ·06705.

Aliter : Since 670·5 = 6·705 × 10^2, therefore

log 670·5 = log 6·705 + 2,

or 2·8264 = log 6·705 + 2,

so that log 6·705 = ·8264.

Now (i) log N = 1·8264 = 1 + log 6·705,
= log (6·705 × 10),
= log 67·05

so that N = 67·05.

(ii) log N = 8·8264 − 10 = − 2 + log 6·705,
= log (6·705 × 10^{-2}),
= log ·06705,

so that N = ·06705.

EXERCISES

1. Find by inspection the characteristic of the logarithm of each of the following numbers. Verify your answers to the problems (vii) to (xii) by writing the numbers in the form $n \times 10^r$, where r is an integer and $1 \leqslant n < 10$.

(i) $2{\cdot}3 \times 10^2$ (ii) $3{\cdot}4 \times 10.$ (iii) $1{\cdot}63 \times 10^4$
(iv) $3{\cdot}97 \times 10^{-1}$ (v) $4{\cdot}72 \times 10^{-5}$ (vi) $2{\cdot}01 \times 10^4$
(vii) 672 (viii) 4329 (ix) 67·87
(x) 4·992 (xi) ·004 (xii) ·0375.

2. Find the characteristic of the logarithm of N when :

(i) log N = 2·0001 (ii) log N = − 3·4621. (iii) log N = − ·0231.
(iv) log N = 3·0632 (v) log N = − 4·0071. (vi) log N = − 6·8927.

3. Given log 6·731 = 0·8281, find the numbers which have the following logarithms :

(i) 2·8281. (ii) 3·8281 (iii) 1·8281.
(iv) 15·8281. (v) 9·8281 − 10. (vi) 8·8281 − 10.

4. Given log 2 = ·3010, find the number of digits in 2^{53}.

5. Given log 3 = ·4771, find the number of zeros preceding the first significant digit in 3^{-31}.

6.7 Table of Logarithms

We have seen above how the characteristic of the common logarithm of any number can be easily written down. The mantissa of the logarithm of any number cannot, however, be determined so easily. By using advanced methods we can compute the mantissa of the logarithm of any number to any desired accuracy.

Table 2 at the end of the book gives the mantissas of logarithms of the integers 100 to 999 The mantissas are, in most cases, non-terminating decimals, and therefore the values in the table are only approximations (except in a few cases). The values in the table are given to four significant figures and therefore the table is called a *four-figure table.*

Let us have a look at the table. The first two digits appear in the first column, headed N. The third digit appears at the top of the page along the first row.

Since the characteristic of the logarithm of any number between 1 and 10 is zero, therefore the table actually gives the logarithms of numbers from 1·00 to 9·99 in steps of ·01. For finding the logarithms of numbers outside this range, we have to express them first in scientific notation and then use the table. We may also have to use interpolation in order to find from the table the logarithm of a number having more than three significant digits. The method will be illustrated by examples.

Example 11. Find the logarithm of

(i) 6·34. (ii) 648. (iii) ·00873..

Solution. (i) The characteristic of $\log 6{\cdot}34 = 0$;
mantissa of $\log 6{\cdot}34$ = mantissa of 634.

To read the mantissa of 634 from the table, we first read down along the first column until we reach 63 and then read to the right until we reach the column headed by 4. We thus get the desired mantissa as ·8021.

$\therefore \log 6{\cdot}34 = 0{\cdot}8021$

(ii) The characteristic of $\log 648 = 2$;
mantissa of long 648 = ·8116.

$\therefore \log 648 = 2.8116.$

(iii) The characteristic of $\log {\cdot}00873 = -3$,
mantissa of $\log {\cdot}00873$ = mantissa of log 873,
$= 9{\cdot}410.$

$\therefore \log {\cdot}00873 = -3 + {\cdot}9410 = 7{\cdot}9410 - 10.$

Aliter: (*i*) The number lies between 1 and 10, and therefore we can read the logarithm directly from the table.

We first read down along the first column until we reach 63 and then read to the right until we reach the column headed by 4. We thus get

$\log 6{\cdot}34 = {\cdot}8021.$

(ii) $648 = 6{\cdot}48 \times 10^2.$

$\therefore \log 648 = \log 6{\cdot}48 + 2,$
$= {\cdot}8116 + 2 = 2{\cdot}8116,$

(iii) $\cdot 00873 = 8{\cdot}73 \times 10^{-3}$.

$\therefore$ $\log \cdot 00873 = \log 8{\cdot}73 + (-3)$.

$= \cdot 9410 + (7 - 10)$,

$= 7{\cdot}9410 - 10$.

Example 12. Find the logarithm of

(i) 3789. (ii) ·06842.

Solution (i) The given number has four significant digits, and its mantissa cannot be read directly from the table. We shall, therefore, have to use interpolation, as explained in Chapter 3, page 91. Let us make the following observations which will help us in the interpolation :

1. The number 3789 is ·9 of the way from 3780 to 3790.

2. The mantissas corresponding to 3780 and 3790 are the same as those corresponding to 378 and 379, which when read from the table are ·5775 and ·5786 respectively. The tabular difference is therefore 11 thousandths.

3. The desired mantissa is ·9 of the way from ·5775 to ·5786. The correction is therefore 11 × ·9 = 9·9 thousandths = 10 thousandths, when rounded off.

4. Since log x increases as x increases, therefore the correction must be added.

5. The desired mantissa = ·5775 + ·0010, = ·5785.

Also, the characteristic = 3.

$\therefore$ log 3789 = 3·5785.

The working for finding the mantissa can be arranged as follows :

(i) Number		Mantissa		Correction
3780		5775		11 × ·9 = 9·9 ≐ 10
3789	9		11	5775 + 10 = 5785.
3790	10	5786		

Since the required characteristic is 3, therefore log 3789 = 3.5785.

(ii) Number		Mantissa		Correction
6840		8351		6 × ·2 = 1·2 ≐ 1
6842	2		6	8351 + 1 = 8352.
6850	10	8357		

Since the required characteristic is —2, therefore
$\log \cdot 06842 = -2 + \cdot 8352 = 8{\cdot}8352 - 10.$

Remark. It is advisable to do all the computations needed to write down the logarithm of a number mentally.

5.8. Antilogarithms

If log N = x, then we say that N is the *antilogarithm* of x. In symbols,

$$\log N = x \Longleftrightarrow N = \text{antilog } x.$$

We can use table 2 to find the antilog of a given logarithm, i.e., to find N when log N is given. The following examples will illustrate the method.

Example 13. Find (i) antilog 1·8722. (ii) antilog (8·6665 − 10).

Solution. (i) Let N = antilog 1·8722, so that log N = 1·8722. The mantissa of log N is ·8722. We look for the entry ·8722 in table 2. We find that it occurs in the line which has 74 in the first column headed by N; also the column in which it occurs is headed by 5. Therefore the succession of digits in N is the same as that in the number 745. Also, the characteristic of log N being 1, there must be two digits to the left of the decimal point.

Hence N = 74.5.

(ii) Let N = antilog (8.6665 − 10), so that log N = 8·6665 — 10. The mantissa of log N is ·6665, so that from table 2 we find that N has the same succession of digits as the number 464. Also, the characteristic of log N being —2 (= 8 — 10), there must be one zero between the decimal point and the first non-zero digit in N.

Hence N = ·0464.

Example. 14. Find (i) antilog 2·9446 (ii) antilog (9·8575 — 10).

Solution. Let N = antilog 2·9446, so that log N = 2·9466. The mantissa of log N is therefore ·9446. On searching through table 2 we find that ·9446 is not an entry, but there are two consecutive entries ·9445 and ·9450 in the row headed by 88 and columns headed by 0 and 1 respectively, between which the given number lies.

Number		Mantissa		Correction
8800	10	9445	1	$10 \times \frac{1}{5} = 2$
?		9446	5	$8800 + 2 = 8802.$
8810		9450		

Since the characteristic of log N is 2, therefore there must be three digits to the left of the decimal point in N.

Hence $N = 880{\cdot}2$.

(ii) Let $N = \text{antilog } (9{\cdot}8575 - 10)$, so that $\log N = 9{\cdot}8575 - 10$. The mantissa of log N is therefore ·8575. From table 2 we find as follows :

Number		Mantissa			Correction
7200		8573			$10 \times \frac{2}{6} = 3\frac{1}{3} = 3$
?	10	8575	2		$7200 + 3 = 7203.$
7210		8579		6	

Since the characteristic of log N is —1, therefore $N < 1$ and there should be no zero between the decimal point and the first non-zero digit in N.

Hence $N = {\cdot}7203$.

EXERCISES

1. Find the logarithm of each of the following numbers :

(i) 4372	(ii) 28710.	(iii) 937600
(iv) 4·639	(v) 24·86	(vi) 123·7.
(vii) ·0614.	(viii) ·000032.	(ix) ·004881.

2. Find the antilogarithm of each of the following logarithms :

(i) 1·4143	(ii) 2·6891.	(iii) 0·8163.
(iv) 9·6438 — 10	(v) 8·4271 — 10	(vi) 3·5376.
(vii) 7·3224 — 10	(viii) 6·8734 — 10.	(ix) 5·9971 — 10.

5.9. Use of Logarithms

Logarithms can be used to compute products, quotients, powers, roots, and combinations of these operations. The following examples will illustrate the procedure.

Example 15. Find the value of

$$\frac{2{\cdot}632 \times {\cdot}0071}{1{\cdot}451 \times {\cdot}0651}.$$

Solution.

$$\text{Let } N = \frac{2{\cdot}632 \times {\cdot}0071}{1{\cdot}451 \times {\cdot}0651}.$$

By taking logarithms of both sides and using theorems 1 and 2, we have

$$\log N = \log (2{\cdot}632 \times {\cdot}0071) - \log (1{\cdot}451 \times {\cdot}0651),$$
$$= (\log 2{\cdot}632 + \log {\cdot}0071) - (\log 1{\cdot}451 + \log {\cdot}0651).$$

In order to avoid errors, it is advisable to prepare an outline of the solution. For the present problem a possible outline can be as follows :

log 2·632 =
log ·0071 =
______________ Add
log numerator =
log denominator =
______________ Subtract
log N =
N =

log 1·451 =
log ·0651 =
______________ Add
log denominator =

By using tables we can complete the above outline. The completed form would be as follows :

log 2·632 = ·4203
log ·0071 = 7·8513 — 10
______________ Add
log numerator = 8·2716 — 10
log denominator = 8·9753 — 10
______________ Subtract
log N = 9·2963 — 10
N = ·1978

log 1·451 = ·1617
log ·0651 = 8·8136 — 10
______________ Add
log denominator = 8·9753 — 10

Example 16. Find the value of $(\cdot 6438)^{1/7}$.

Solution. Let $N = (\cdot 6438)^{1/7}$.

$$\text{Then } \log N = \tfrac{1}{7} \log (\cdot 6438),$$
$$= \tfrac{1}{7} (9{\cdot}8087 - 10).$$

If we divide 9·8087 — 10 by 7 we get the result in the form 1·4012 .... — 1·4285 .... which is not suitable for logarithmic computation. We can, however, obtain the result in a suitable form if we write the characteristic 9 — 10 as 69 — 70 before division. We then get

$$\log \cdot 6438 = 69{\cdot}8087 - 70$$
$$\tfrac{1}{7} \log \cdot 6438 = 9{\cdot}9727 - 10$$
$$N = \cdot 9391.$$

Example 17. Compute $\dfrac{(27{\cdot}43)^{1/2} (\cdot 6804)^{1/3}}{(7{\cdot}091)^{1/4}}$.

Solution. Let $N = \dfrac{(27{\cdot}43)^{1/2} (\cdot 6804)^{1/3}}{(7{\cdot}091)^{1/4}}$.

$$\text{Then } \log N = \{(\tfrac{1}{2} \log)\ 27{\cdot}43 + \tfrac{1}{3} \log \cdot 6804)\} - \tfrac{1}{4} \log (7{\cdot}091).$$

Preparing an outline and filling the numbers we have

$\log 27{\cdot}43 = 1{\cdot}4383$

$\log {\cdot}6804 = 29{\cdot}8326 - 30$

$\frac{1}{2}\log 27{\cdot}43 = {\cdot}71915$

$\frac{1}{3}\log {\cdot}6804 = 9{\cdot}9442 - 10$

A

$\log \text{numerator} = {\cdot}6634$ $\qquad \log 7{\cdot}091 = {\cdot}8507.$

$\log \text{denominator} = {\cdot}2127. \longleftarrow (\frac{1}{4}) \log 7{\cdot}091 = {\cdot}2127.$

S

$\log N = {\cdot}4407$

$N = 2{\cdot}758.$

Example 18. Compute $\dfrac{(27{\cdot}43)^{1/2}\,(-{\cdot}6804)^{1/3}}{(7{\cdot}091)^{1/4}}$.

Solution. We have not defined log x when x is negative. Therefore we cannot directly apply logarithms for computing the value of the given expression. However, $(-{\cdot}6804)^{1/3} = -({\cdot}6804)^{1/3}$.

Therefore the given expression $= -\dfrac{(27{\cdot}43)^{1/2}\,({\cdot}6804)^{1/3}}{(7{\cdot}091)^{1/4}}$

$= -N$, say.

We may evaluate N by using logarithms and then affix a negative sign to it to obtain the value of the given expression.

By the preceding example, $N = 2{\cdot}758$.

Hence the given expression $= -2{\cdot}758$.

EXERCISES

Compute by means of logarithms :

1. $28{\cdot}34 \times 0{\cdot}8176$.
2. $(4{\cdot}315)^3$.
3. $(2839)^{1/4}$.
4. $(128{\cdot}6)^{1/2} \times (29{\cdot}74)^{1/3}$.
5. $\dfrac{2{\cdot}962 \times {\cdot}04725}{24{\cdot}35}$.
6. $(4{\cdot}973)^{-\frac{5}{2}}$.
7. $\sqrt{\left\{\dfrac{1{\cdot}287 \times 4{\cdot}682}{3{\cdot}421}\right\}}$.
8. $\sqrt{\left\{\dfrac{6{\cdot}842 \times 27{\cdot}04}{{\cdot}03145}\right\}}$.
9. $\sqrt{\left\{\dfrac{4{\cdot}82 \times ({\cdot}06217)}{28{\cdot}13}\right\}}$.
10. $\dfrac{26{\cdot}34}{\sqrt[3]{7{\cdot}63} \times 0{\cdot}634}$.
11. $\dfrac{\sqrt[4]{605} \times 8{\cdot}973}{\sqrt{7563}}$.
12. $\dfrac{(4{\cdot}218)^{0{\cdot}2} \times \sqrt{6{\cdot}87}}{12{\cdot}92}$.

13. $\dfrac{(4{\cdot}96)^{0{\cdot}3} \times (0{\cdot}935)^{0{\cdot}2}}{(0{\cdot}9762)^{0{\cdot}4}}$.

14. $\dfrac{(28{\cdot}14)^{2/3}}{0{\cdot}07359 \times (3{\cdot}82)^{1/4}}$.

15. $\dfrac{(6{\cdot}925)^{3/4} \times \sqrt[3]{74{\cdot}18}}{\sqrt{8{\cdot}432}}$.

16. $\dfrac{42{\cdot}81}{(27{\cdot}63)^{0{\cdot}3} \times \sqrt[3]{1{\cdot}734}}$.

17. $\dfrac{(28{\cdot}45)^{1/2} \times (-7{\cdot}621)^{1/3}}{(4{\cdot}872)^{1/4}}$.

18. $\dfrac{(17{\cdot}61)^{1/2} \times (2{\cdot}624)^{1/5}}{(-0{\cdot}0473)^{1/3}}$.

19. $\sqrt[3]{97{\cdot}63} - \sqrt{12{\cdot}83}$.

20. $(62{\cdot}59)^{1/4} - (76{\cdot}41)^{1/5}$.

5.10. Exponential Equations

An equation of the form

$$a^x = b,$$

that is, an equation in which the unknown appears as an exponent is called an *exponential equation*. To solve an exponential equation we have to first convert it into logarithmic form.

Example 19. Solve : $7^x = 1{\cdot}69$.

Solution. Equating the logarithms of the two members of the given equation, we have

$$\begin{aligned} \log 7^x &= \log 1{\cdot}69, \\ \text{or } x \log 7 &= \log 1{\cdot}69. \\ \text{Hence} \quad x &= \frac{\log 1{\cdot}69}{\log 7}, \\ &= \frac{{\cdot}2279}{{\cdot}8451}. \end{aligned}$$

The problem now is to compute the value of the above fraction

$$\begin{aligned} \log x &= \log \frac{{\cdot}2279}{{\cdot}8451}. \\ &= \log {\cdot}2279 - \log {\cdot}8451. \\ &= (9{\cdot}3577 - 10) - (9{\cdot}9270 - 10), \\ &= 9{\cdot}4307 - 10. \\ \therefore x &= \text{antilog } (9{\cdot}4307 - 10), \\ &= {\cdot}2696. \end{aligned}$$

EXERCISES

1. Solve each of the following equations for x :

(i) $7^x = 25$.

(ii) $100^x = 27$.

(iii) $(3/2)^x = 4/3$.

(iv) $3^{x2} = 64$.

(v) $3^{x+2} = 9{\cdot}76$.

(vi) $10^{x+1} = 298{\cdot}7$.

(vii) $(6{\cdot}071)^x = 1000$.

(viii) $(27.32)^x = 25.4$.

(ix) $5.\ 6^x = 7.\ 5^x$.

(x) $(3{\cdot}271)^{2x} (4{\cdot}819)^{x+1} = 278$.

(xi) $2^{2x} - 8.\ 2^x + 15 = 0.$ (xii) $3^{2x} - 7.3^x + 10 = 0.$

(xiii) $5^{2x} - 5^{x+1} + 6 = 0.$ (xiv) $49^x + 7^x - 6 = 0.$

2. Solve each of the following inequations :

(i) $2^x > 7.$ (ii) $3\,(2^x) \leqslant 5.$

(iii) $(3/4)^x > \cdot 99.$ (iv) $(1{\cdot}8)^x \geq 3{\cdot}6.$

(v) $2^{-x} < \cdot 2.$ (vi) $4^x < \cdot 62.$

3. Find the following logarithms :

(i) $\log_2 63{\cdot}84.$ (ii) $\log_3 41{\cdot}96.$

5.11. Applications of Logarithms to Problems

Example 20. A certain bank pays 6 per cent per annum interest on fixed deposits. How much will Rs. 4,000 amount to in 15 years, interest being compounded annually?

Solution. If the required amount be A rupees, then

$$A = 4000\,(1 + \cdot 06)^{15}.$$

By taking logarithms of both sides we have,

$$\begin{aligned} \log A &= \log 4000 + 15 \log 1{\cdot}06, \\ &= 3{\cdot}6021 + 15 \times \cdot 0253, \\ &= 3{\cdot}6021 + \cdot 3795, \\ &= 3{\cdot}9816, \\ \text{so that } A &= \text{antilog } 3{\cdot}9816, \\ &= 9585. \end{aligned}$$

Hence the required amount is Rs. 9585, correct to the nearest rupee.

Example 21. Radium decays according to the equation

$$Q = Q_o\, 2^{-\frac{t}{1600}},$$

where Q_o is the initial mass of radium and Q is the mass of radium after t years. If the initial mass be 450 milligrams, in how many years will the mass be cut down to 300 milligrams ? In other words, find the one-third life of radium.

Solution. We are given that $Q_o = 450$, $Q = 300$.

We have, therefore, to solve the equation

$$300 = 450.\ 2^{-\frac{1}{1600}t}.$$

The above equation is equivalent to the equation

$$\frac{2}{3} = 2^{-\frac{1}{1600}t}.$$

Taking logarithms of both sides, we have

$$\log 2 - \log 3 = -\frac{t}{1600}\log 2,$$

$$\text{or} \quad \cdot3010 - \cdot4771 = -\frac{t}{1600} \times \cdot3010,$$

$$\text{or} \quad t = \frac{\cdot1761 \times 1600}{\cdot3010}.$$

By taking logarithms, we have

$$\begin{aligned} \log t &= \log \cdot1761 + \log 1600 - \log \cdot3010, \\ &= (9\cdot2458 - 10) + 3\cdot2041 - (9\cdot4786 - 10), \\ &= 2\cdot9713. \end{aligned}$$

$$\begin{aligned} \text{Therefore} \quad t &= \text{antilog } 2\cdot9713 \\ &\doteq 936. \end{aligned}$$

Thus it will take approximately 936 years for 450 milligrams to decay to 300 milligrams.

EXERCISES

1. If Rs. 5,000 is invested at 5 per cent interest compounded annually, what will be the amount in the investment after 10 years ?
2. A certain bank pays 4 per cent interest compounded half-yearly. How much will Rs. 4,000 amount to if left on deposit for 15 years ?
3. A certain bank pays 6 per cent interest compounded quarterly. What will Rs. 2,500 amount to if kept in the bank for 12 years ?
4. How much should be invested now at 5 per cent compounded half-yearly to amount Rs. 975 in 8 years ?
5. If energy E and mass m are related by the equation $E = mc^2$, where c is the velocity of light ($= 3{\cdot}00 \times 10^{10}$ cm per sec.), find the energy available in a mass of 4 gm.

Note : When mass is expressed in grammes, and velocity is expressed in centimetres per second, then energy is expressed in ergs.

6 With the same notation as in the above problem, find in kilowatt hours, the energy available in a mass of 4·5 gm. Remember that 1 kilowatt hour $= 3{\cdot}60 \times 16^6$ joules, and 1 poule $= 1{\cdot}00 \times 10^7$ ergs.

7. A certain culture of bacteria (under ideal conditions) grows at the rate of $N = 1100\,(10)^{t/45}$; where N is the number of bacteria, and t is the elapsed time in hours. How many bacteria will be present after 5 hours ?

8. A certain radioactive material decays at a rate given by $Q = Q_o\,(10)^{-0{\cdot}025t}$ where t is in hours. Find the half-life of the material, i.e., find t when

$$Q = \tfrac{1}{2}\,Q_o.$$

9. A certain radioactive substance decays at a rate given by $Q = Q_o\,10^{-kt}$ where Q is in grams and t is in years. If $Q_o = 1000$ gm, find k if $Q = 800$ gm, when $t = 1000$ years.

10. The mass of a quantity of radium steadily decreases because of loss by radiation. It is known that the quantity Q of radium remaining after t years is given by $Q = Q_o\,2^{-t/1600}$, where Q_o is the initial quantity of radium. If the initial quantity is 750 milligrams, in how many years will it reduce to 250 milligrams ?

11. A general formula for exponential growth and decay is $Q = Q_o\,10^{kt}$ where Q is the quantity at time t and Q_o is the quantity initially.
    (i) If the formula is a growth formula, what is the sign of k ?
    (ii) If the formula is a decay formula, what is the sign of k ?
    (iii) Find k for a culture which doubles in 3 hours, t being measured in hours.
    (iv) Find k for a radioactive element with a half-life of 200 years, t being measured in years.

12. The period of vibration of a simple pendulum whose length is 1 centimetre is given by $T = 2\pi\left(\frac{1}{g}\right)^{\frac{1}{2}}$. Taking $g = 981$, find the time of vibration of a pendulum 40 centimetres in length.

13. The volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$. Find the volume of a a hollow sphere having outer radius $7\frac{1}{2}$ cm and inner radius 5 cm.

14. A change of state of air is described by the pressure-volume relation $P V^{7/5} = p_o\, v_o^{\,7/5}$.

    Find P if $p_o = 321$, v_o $75.32 =$ and $V = 47{\cdot}81$.

15. If the temperature of a rod of length l_o is increased by t degrees, it expands so that its final length is

$$l = l_o\,(1 + \alpha t),$$

where α is a constant depending on the material of which the rod is constructed. For steel, $\alpha = {\cdot}00001071$, approximately. Steel rails, 9 metres in length, are spaced 3/4 of a centimetre apart. Is this sufficient if the rails are subject to a variation in temperature 92° ?

5.12. Logarithms of Trigonometric Functions

In solving problems involving trigonometric functions with the help of logarithms we have to use logarithms of the values of trigonometric functions. Such situations will arise very often while solving triangles. We can find the logarithm of the value of a trigonometric function of an angle by first finding the value of the trigonometric function of the angle from table 1 and then finding the logarithm of this value from table 2. However, we need not take the trouble of consulting two tables because there are tables from which the logarithm of the value of a trigonometric function of an angle can be found directly. Table 3 at the end of the book gives the logarithms of the sine, cosine, tangent and cotangent of angles from 0° to 90°, at intervals of $10'$. Since $\log \sec \theta = - \log \cos \theta$, and $\log \operatorname{cosec} \theta = -\log \sin \theta$, therefore this table also suffices to find out $\log \sec \theta$ and $\log \operatorname{cosec} \theta$ whenever θ lies between 0° and 90°.

The following examples will illustrate the method of using the table.

Example 22. Find $\log \cos 55^\circ\ 32'$.

Solution. From table 3 we have the following readings :

$$\log \cos 55^\circ\ 30' = 9{\cdot}7513 - 10.$$
$$\log \cos 55^\circ\ 40' = 9{\cdot}7513 - 10.$$

The tabular difference is .0018. Now $55^\circ\ 32'$ is 2/10 of the way from $55^\circ\ 30'$ to $55^\circ\ 40'$. Assuming that the same relation holds good between $\log \cos 55^\circ\ 32'$, $\log \cos 55^\circ\ 30'$ and $\log \cos 55^\circ\ 40'$, we find that the correction $= {\cdot}0018 \times \frac{2}{10} = {\cdot}0004$. Since $\cos \theta$ decreases as θ increases from 0° to 90°, therefore $\log \cos \theta$ decreases as θ increases from 0° to 90°. Consequently the correction must be subtracted.

$$\text{Therefore } \log \cos 55^\circ\ 32' = (9{\cdot}7531 - 10) - {\cdot}0004,$$
$$= 9{\cdot}7527 - 10.$$

Example 23. Find the acute angle θ for which $\log \sin \theta = 9{\cdot}6301 - 10$.

Solution. From table 3 we find that while $9{\cdot}6301 - 10$ is not an entry in the column headed by log sin, but we find that

$$\begin{tikzcd}
\log \sin 25^\circ\ 10' = 9{\cdot}6286 - 10 \arrow[d, bend left, "15"] \arrow[dd, bend left=60, "27"] \\
\log \sin \quad ? = 9{\cdot}6301 = 10 \\
\log \sin 25^\circ\ 20' = 9{\cdot}6313 - 10
\end{tikzcd}$$

Now $9{\cdot}6301 - 10$ is $\frac{15}{27}$ of the way from $9{\cdot}6286 - 10$ to $9{\cdot}6313 - 10$. Assuming that the same holds good between the angles, we find that the correction $= 10' \times \frac{15}{27} = (5\frac{5}{9})' = 6'$.

Since θ and $\sin \theta$ increase together in the first quadrant, therefore θ and $\log \sin \theta$ also increase together in the first quadrant, and consequently the correction must be added.

Hence $\theta = 25^\circ\ 10' + 6' = 25^\circ\ 16'$.

EXERCISES

1. Find, by using table 3, the value of each of the following :

(i) log sin 31° 40′.	(ii) log cos 68° 10′.
(iii) log tan 51° 27′.	(iv) log cot 23° 39′.
(v) log sec 48° 46′.	(vi) log csc 51° 13′.
(vii) log sin 26° 55′.	(viii) log cos 18° 29′.
(ix) log tan 54° 19′.	(x) log cot 39° 32′.

2. Find by using table 3, the acute angle θ to the nearest minute for each of the following :

(i) $\log \sin \theta = 9{\cdot}4269 - 10$.	(ii) $\log \cos \theta = 9{\cdot}9602 - 10$.
(iii) $\log \tan \theta = 9{\cdot}3149 - 10$.	(iv) $\log \cot \theta = {\cdot}3916$.
(v) $\log \sec \theta = {\cdot}0162$.	(vi) $\log \csc \theta = {\cdot}3679$.
(vii) $\log \sin \theta = 9{\cdot}8187 - 10$.	(viii) $\log \cos \theta = 9{\cdot}8613 - 10$.
(ix) $\log \tan \theta = 9{\cdot}7263 - 10$.	(x) $\log \cot \theta = {\cdot}0421$.

ASSORTED EXERCISES ON CHAPTER 5.

1. Using $\pi = 3.14$, approximate 3^{π}.

2. Solve the equation $\log_{10}(x + 2) - \log_{10} x = {\cdot}6281$.

3. Solve the inequation $(2{\cdot}5)^x \leqslant 6{\cdot}4$.

4. Find to four significant digits $\log_{2\cdot718} 10$.

5. Find :

(i) log (log 3).	(ii) (log 2) ÷ (log 6).
(iii) (log 3) (log 4).	(iv) $(\log 3)^3$.

6. Find in how much time will a sum of money double itself if $i = {\cdot}05$, the interest being compounded annually.

7. Solve for x :

$$\log_3 |3x + 1| - \log_3 |2x + 4| = 1.$$

8. A radioactive substance decays according to equation $Q = Q_o (10)^{-0\cdot026\,t}$, where t is in hours. Find the half-life of the substance.

9. Find the value of $(-97{\cdot}26)^{3/5}$.

10. If a and b be positive real numbers such that $a^2 + b^2 = 7ab$, prove that

$$\log \frac{a + b}{3} = \tfrac{1}{2}(\log a + \log b).$$

11. If $a > 0$, $x = \log_{2a} a$, $y = \log_{3a} 2a$, $z = \log_{4a} 3a$, prove that $xyz + 1 = 2yz$.

12. If a and b be positive numbers other than unity, prove that

$$\log_a x . \log_b y = \log_b x . \log_a y.$$

13. Find the value of $(6{\cdot}4)^{1/10} \times (\sqrt[4]{\cdot 256})^3 \div \sqrt{80.}$

14. Find the seventh root of $\cdot 03457$.

15. Find the number of digits in 2^{61}.

16. Find the position of the first significant digit in 3^{-38}.

17. If $a^{1/3} = x^{1/3} + y^{1/3}$, find the value of a when $x = 0{\cdot}3753$, $y = 0{\cdot}01984$.

18. In the formula $V = 3{\cdot}142\ r^2 h$, find the value of V when $r = 6{\cdot}39$, $h = 12{\cdot}87$

19. Prove that $\log \dfrac{3x-\sqrt{9x^2-1}}{3x+\sqrt{9x^2-1}} = 2 \log (3x - \sqrt{9x^2-1})$.

20. Solve each of the following equations :
    (i) $(\log x)(\log x^2) + \log x^3 = 2$.
    (ii) $\log(\log x) = 2$.

21. If the birth rate of a nation is 11 per cent and death rate 3 per cent per year, how many years (to the nearest integer) will it take for the population to double ?

22. Evaluate to four significant digits :

$$\frac{\sqrt[7]{8400}}{(-38{\cdot}97)\ (2{\cdot}165)^3}.$$

23. If $c^2 = a^2 + b^2$, is 2 log c equal to 2 log a plus 2 log b ?

24. Prove that $2(\log 2 + \log p + \log q)$

$$= \log[(p^2 + q^2)^2 - (p^2 - q^2)^2],$$ if p and q are positive.

25. A nautical mile is defined as being equal to one minute of arc on the surface of the earth at the equator. If the radius of the equatorial circle is 6349 km, the number of metres in a nautical mile is therefore

$$\frac{2\pi \times 6349 \times 1000}{360 \times 60}.$$

Make the calculation, taking $\pi = 3{\cdot}142$, and obtain your result correct to four significant digits.

Chapter 6

Solution of Triangles and Its Applications

6.1. Introduction

The three sides and the three angles of a triangle are called the parts of the triangle. Given the measures of three parts, at least one of which is a side, we can compute the possible measures of the remaining parts. The process of computing the measures of the unknown parts from those of the given parts is known as *solving* a triangle. Since we are going to be interested only in the measures of the parts of a triangle, therefore we shall not distinguish between congruent triangles. Also, we shall use the phrase 'a part of a triangle' to mean the measure of that part. We shall say that three given parts determine a triangle if there exists a triangle with given parts and if all the other parts can be computed uniquely by means of the given parts. Thus, for example, the three sides determine a triangle provided no side is as great as the sum of the other two; three angles do not determine a triangle because if the sum of the angles be different from 180°, then no triangle exists with the given angles as parts, and if the sum of the angles be 180°, then infinitely many triangles exist with the given angles as parts.

Solution of triangles has lots of applications in navigation, astronomy, surveying, architecture, engineering, and in many other sciences. In astronomy, the radius of the earth, the radius of the moon and the distance of the moon from the earth are computed by the process of solving triangles. In surveying (and elsewhere too) heights and distances which cannot be measured directly are computed by solving triangles. In fact, till the beginning of the present century, the study of trigonometry was motivated solely by its utility in solving triangles. (Unfortunately, most textbooks still cling to this practice !) In modern times, however, the trigonometric functions have found important applications in numerous other fields such as simple harmonic motion, the theory of vibrating strings, Fourier series, etc., and this has led the mathematicians to emphasize the analytical aspects of trigonometry more than the computational aspect. Even then, solution of triangles continues to be sufficiently important.

In the present chapter we propose to discuss various methods of solving triangles and their elementary applications to heights and distances, navigation, etc. We shall divide our study into three parts—solution of right-angled triangles, solution of oblique triangles, and applications. However, we shall first set out some general directions which will prove useful while solving triangles.

Note : We shall throughout use the same notation as in chapter 4.

6.2. Some General Directions

The following points should be remembered while solving a triangle:

1. By using the given parts, construct a triangle roughly to scale. Mark the known parts of the triangle and indicate the unknown parts. A proper figure helps in having a clearer understanding of the problem and a better planning of the solution. It guides us very much in certain cases (for example in case IV, p. 159). The figure also gives a rough estimate of the parts required to be computed and at the same time it serves as a check against gross errors.
2. While solving triangles, we must remember that the given parts are generally the result of certain measurements and, as such, are only approximate quantities Also, the computed parts cannot be any more accurate than the given parts. We have therefore to see as to what degree of accuracy in the computed parts can possibly be there when the given parts have a certain degree of accuracy. A computed side can be accurate to at the most as many significant digits as the given side or sides contain. Therefore we should always round off a computed side to as many significant digits as the given side or sides contain. Similarly, a computed angle can be accurate to the nearest degree, nearest multiple of 10′, or nearest minute at the most, according as the given angle or angles are accurate to the nearest degree, the nearest multiple of 10′ or nearest minutes.

Since in any problem angles and sides are involved together, therefore, we should know as to what accuracy in angles is comparable to a given accuracy in the sides. The following table gives the accuracy in an angle (or angles) corresponding to a given accuracy in a side (or sides).

Sides	Angles
Two significant digits	Nearest degree
Three significant digits	Nearest multiple of 10′
Four significant digits	Nearest minute
Five significant digits	Nearest tenth of a minute

According to the above table, which we shall adhere to, a computed angle should be rounded off to the nearest degree, nearest multiple of 10′, nearest minute or nearest tenth of a minute according as the sides have two, three, four or five significant digits. We shall throughout assume that the given parts of a triangle to be solved are of corresponding accuracies.

3. Solution of triangles involves a lot of multiplications and divisions which may be done either without any computational aids, or we may choose to use a slide rule, logarithmic tables or calculating machines to perform the desired computations. If the results are required only to two significant digits, then we need not use any computational aids. If the results are required to not more than three significant digits, we may use the slide rule. If a higher degree of accuracy is desired, then we may use logarithm tables. Calculating machines give most accurate result, but we do not generally have access to them and therefore we shall not discuss their use for solution of triangles.

   The slide rule can also be used as a check on the working when other methods are used for computation, unless of course the answers are to be more accurate than the slide rule can give.

4. For each part to be computed, choose a formula which determines that part. The choice of the formula will generally depend on the mode of computation. Different formulae may be needed according as we wish to use logarithms or we wish to do without them.

5. It is always better, if possible, to find a part from the given parts rather than from the computed parts. This gives better results and also avoids carrying forward any error in computation.

6. It is extremely important to check the results in problems on solution of triangles as elsewhere too. While a figure to scale serves as a rough check and the use of the slide rule gives a check when a high degree of accuracy is not desired, the solution can always be checked either by computing a part by two different methods or by substituting the parts in a formula which holds for triangles and which has not been used in the process of solution. While using

the latter method for checking the solutions, the following formulae are particularly useful :

$$\text{(i)}\quad \frac{a+b}{c} = \frac{\cos\frac{1}{2}(A-B)}{\sin\frac{1}{2}C} \qquad \text{(ii)}\quad \frac{a-b}{c} = \frac{\sin\frac{1}{2}(A-B)}{\cos\frac{1}{2}C}\,.$$

The above formulae, known as Molleweide's formulae, have the redeeming feature that each of them involves all the six parts of a triangle. (The derivation of these formulae is left as an exercise for the reader.)

7. The working should be arranged in a neat and systematic manner. A complete outline of the working should be prepared before doing any computations.

6.3. Solution of Right-angled Triangles

If one of the angles of a triangle is given to be right angle, then the triangle can be solved easily without having recourse to the formulae derived in Chapter 4.

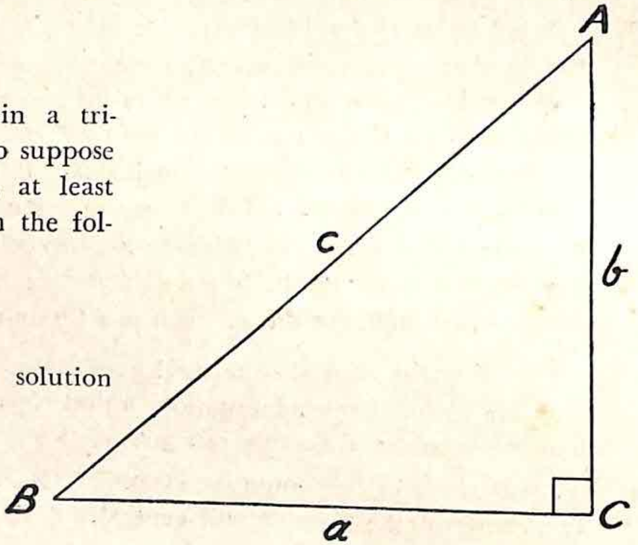


Fig. 6.1

Suppose it is known that in a triangle ABC, $\angle C = 90°$. Let us also suppose that two other parts are given, at least one of them being a side. Then the following different cases arise:

Case I : Given a and b. The solution is unique.

We use the formulae

$\tan A = a/b$, to find A;

$B = 90° - A$, to find B ;

$$\left.\begin{array}{ll} & c = b/\cos A, \text{ to find } c, \\ \text{or} & c = \sqrt{(a^2 + b^2)}, \text{ to find } c. \end{array}\right\}\text{(Use whichever is simpler.)}$$

Case II : Given a and c. There is no solution unless $c > a$.

If $c > a$, there is a unique solution.

We use the formulae

$\sin A = a/c$, to find A ;

$B = 90° - A$, to find B ;

$$\left.\begin{array}{ll} & b = a \cot A, \text{ to find } b, \\ \text{or} & b = \sqrt{(c^2 - a^2)}, \text{ to find } b. \end{array}\right\}\text{(Use whichever is simpler.)}$$

Case III : Given a and A. There is no solution unless A is acute. If a is acute, there is a unique solution.

We use the formulae

$$\begin{array}{ll} & B = 90^\circ - A, \text{ to find } B\,; \\ & b = a \cot A, \text{ to find } b\,; \\ & \left.\begin{array}{l} c = a/\sin A, \text{ to find } c, \\ c = \sqrt{(a^2 + b^2)}, \text{ to find } c. \end{array}\right\} \text{(Use whichever is simpler.)} \\ \text{or} & \end{array}$$

Case IV : Given c and A. There is no solution unless A is acute. If A is acute, there is a unique solution.

We use the formulae

$$\begin{array}{ll} & B = 90^\circ - A, \text{ to find } B\,; \\ & a = c \sin A, \text{ to find } a\,; \\ & \left.\begin{array}{l} b = c \cos A, \text{ to find } b\,; \\ b = \sqrt{(c^2 - a^2)}, \text{ to find } b. \end{array}\right\} \text{(Use whichever is simpler.)} \\ \text{or} & \end{array}$$

Example 1. Given $a = 166{\cdot}1$, $c = 187{\cdot}3$, $C = 90^\circ$, solve the triangle.

Solution.

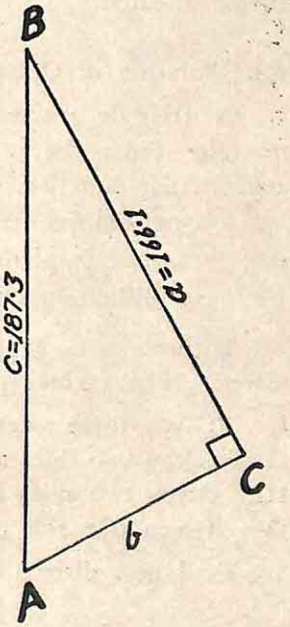


Fig. 6.2

$$(1)\ \sin A = \frac{a}{c} = \frac{166{\cdot}1}{187{\cdot}3}$$

Preparing an outline of the logarithmic solution and filling in the numbers we have

$$\begin{array}{rl} \log 166{\cdot}1 &= 2{\cdot}2204 \\ \log 187{\cdot}3 &= 2{\cdot}2725 \\ \hline & \text{S} \\ \log \sin A &= 9{\cdot}9479 - 10 \\ A &= 62^\circ\, 30'. \end{array}$$

(2) $B = 90^\circ - A = 90^\circ - 62^\circ\, 30' = 27^\circ\, 30'$.

(3) $b = a \cot A = 166{\cdot}1 \cot 62^\circ\, 30'$

$$\begin{array}{rl} \log 166{\cdot}1 &= 2{\cdot}2204 \\ \log \cot 62^\circ\, 30' &= 9{\cdot}7165 - 10 \\ & \underline{\hspace{4cm}}\,A \\ \log b &= 1{\cdot}9369 \\ b &= 86{\cdot}48. \end{array}$$

The computed parts are $A = 62^\circ\, 30'$, $B = 27^\circ\, 30'$, $b = 86.48$.

Check. To check the solution we may compute b by the formula $b = c \cos A = 187.3 \cos 62^\circ\, 30'$.

$$\begin{array}{rl} \log 187{\cdot}3 &= 2{\cdot}2725 \\ \log \cos 62^\circ 30' &= 9{\cdot}6644 - 10 \\ & \underline{\hspace{4cm}}\,A \\ \log b &= 1{\cdot}9369 \end{array}$$

$\therefore\quad b = 86{\cdot}48$, which agrees with the value of b found by another method.

6.4. Solution of Oblique Triangles

A triangle, none of whose angles is a right angle, is called an *oblique triangle.* Since the ratio of two sides of an oblique triangle does not represent any circular function of an angle of triangle, therefore certain formulae are needed for solving oblique triangles. The formulae needed are the cosine formulae, the sine formulae, Napier's formulae and the half-angle formulae, all of which have been established in chapter 4.

If three parts, at least one of which is a side, be given, the triangle can be solved. The various possibilities can be divided into the following four cases :

I. Given three sides (SSS) ;
II. Given two sides and the included angle (SAS) ;
III. Given two sides and the angle opposite to one of them (SSA) ;
IV. Given one side and two angles (SAA).

We shall now discuss each of these cases separately.

6.5. Case I. Three Sides Given (SSS)

To solve a triangle ABC when a, b and c are given. There will be no solution unless the sum of each pair of sides exceeds the remaining side. There will be a unique solution if the sum of each pair of sides is greater than the third. The angles can be found by using the cosine formulae.

$$\text{Cos A} = \frac{b^2 + c^2 - a^2}{2bc},$$

$$\text{Cos B} = \frac{c^2 + a^2 - b^2}{2ca},$$

$$\text{Cos C} = \frac{a^2 + b^2 - c^2}{2ab}.$$

The above formulae are, however, not convenient and therefore used only if the solution is to be obtained without the aid of logarithms.

For logarithmic computation the following half-angle formulae are useful:

$$\tan A/2 = \sqrt{\left\{\frac{(s-b)(s-c)}{s(s-a)}\right\}},$$

$$\tan B/2 = \sqrt{\left\{\frac{(s-c)(s-a)}{s(s-b)}\right\}},$$

$$\tan C/2 = \sqrt{\left\{\frac{(s-a)(s-b)}{s(s-c)}\right\}}.$$

We can determine A/2, B/2, C/2 from the above formulae and A, B, C can be determined therefrom. In practice, however, it is most convenient to use the following versions of the above formulae.

$$r = \sqrt{\left\{\frac{s-a)(s-b)(s-c)}{s}\right\}}$$

$$\tan A/2 = r/(s-a),\ \tan B/2 = r/(s-b),\ \tan C/2 = r/(s-c).$$

Example 3 illustrates the procedure.

The solution can also be obtained without using any formulae as in example 4. Whichever method we may use, the solution is checked by finding A + B + C and examining its deviation from 180°.

Example 2. Solve the triangle ABC with $a = 4,\ b = 7,\ c = 9$.

Solution. $a^3 = 16,\ 2ab = 58,$
$b^2 = 49,\ 2bc = 126,$
$c^2 = 81,\ 2ca = 72,$

so that

$$\left.\begin{aligned} \text{Cos A} &= \frac{b^2 + c^2 - a^2}{2bc} = \cdot 9048, \text{ or } A = 25^\circ, \\ \text{Cos B} &= \frac{c^2 + a^2 - b^2}{2ca} = \cdot 6667, \text{ or } B = 48^\circ, \\ \text{Cos C} &= \frac{a^2 + b^2 - c^2}{2ab} = -\cdot 2859, \text{ or } C = 106^\circ, \end{aligned}\right\} \text{ to the nearest degree.}$$

Check : $A + B + C = 25^\circ + 48^\circ + 106^\circ = 179^\circ$.

The check is satisfactory. The deficit of 1° in the sum is not at all surprising since the value of each angle has been computed to the nearest degree.

Example 3. In a triangle ABC, $a = 210$, $b = 218$, $c = 360$. Solve the triangle.

Solution. We have to first compute s, s−a, s−b and s−c. We can then obtain log r from these values by using the formula

$$\log r = 1/2 \{\log(s - a) + \log(s - b) + \log(s - c) - \log s\}.$$
$$s = \tfrac{1}{2}(a + b + c) = \tfrac{1}{2}(210 + 218 + 360) = 394.$$

$$\begin{aligned} s - a &= 184, \\ s - b &= 176, \\ s - c &= 34. \end{aligned}$$

$$\begin{aligned} \log(s - a) &= 2{\cdot}2648 \\ \log(s - b) &= 2{\cdot}2455, \\ \log(s - c) &= \underline{1{\cdot}5315}\ \text{Add} \\ & \ 6{\cdot}0418 \\ \log\ s &= \underline{2{\cdot}5955}\ \text{Subtract} \\ 2 \log r &= 3{\cdot}4463 \\ \log r &= 1{\cdot}7232 \text{ (approx.)} \end{aligned}$$

$$\begin{aligned} \tan 1/2A &= \frac{r}{s - a} \\ \log r &= 11.7232 - 10 \\ \log(s-a) &= \underline{2.2648}\ \text{S} \\ \log \tan 1/2A &= 9.4584 - 10 \\ 1/2A &= 16^\circ \\ A &= 32^\circ \end{aligned}$$

$$\begin{aligned} \tan B/2 &= \frac{r}{s - b} \\ \log r &= 11{\cdot}7232 - 10 \\ \log(s - b) &= \underline{2{\cdot}2455}\ \text{S} \\ \log \tan B/2 &= 9{\cdot}4777 - 10 \\ B/2 &= 16^\circ\ 40' \\ B &= 33^\circ\ 20' \end{aligned}$$

$$\begin{aligned} \tan C/2 &= \frac{r}{s - c} \\ \log r &= 11{\cdot}7232 - 10 \\ \log(s - c) &= \underline{1{\cdot}5315}\ \text{S} \\ \log \tan C/2 &= 10.1917 - 10 \\ C/2 &= 57^\circ\ 20' \\ C &= 114^\circ\ 40'. \end{aligned}$$

Check : $A + B + C = 32^\circ + 33^\circ\ 20' + 114^\circ\ 40' = 180^\circ$.

Example 4. Solve the triangle ABC when a = 20, b = 17, c = 27.

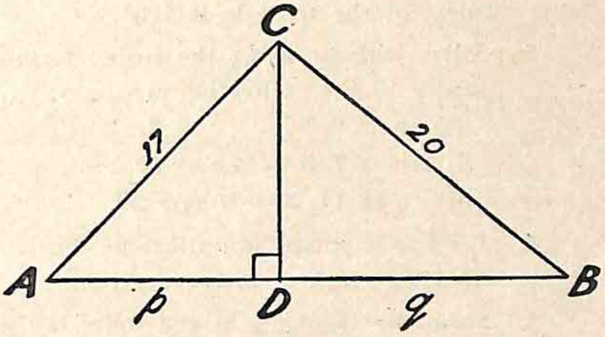


Fig. 6.3

Solution Through C draw $CD \perp AB$.

Let AD = p, DB = q.

Then $p + q = AD + DB$

$= AB = c = 27.$ ......(i)

Also from the right-angled triangles ADC and CDB we have

$$CD^2 = (17)^2 - p^2 = (20)^2 - q^2,$$

or

$$q^2 - p^2 = (20)^2 - (17)^2 = 111. \qquad \text{............(ii)}$$

From (i) and (ii) we have

$$q - p = \frac{37}{3}. \qquad \text{............(iii)}$$

From (i) and (iii) we get $p = \frac{103}{9}$, $q = \frac{140}{9}$. ....(iv)

From the right-angled triangle ADC, we have

$$\text{Cos } A = \frac{p}{17} = \frac{103}{153} = \cdot 6732,$$

so that $\angle A = 47° \, 40'$, to the nearest 10′.

Again, from the right-angled triangle CDB, we have

$$\text{Cos } B = \frac{q}{20} = \frac{7}{9} = \cdot 7778,$$

so that

$\angle B = 39°$, to the nearest 10′,

$\angle C = 180° - (\angle A + \angle B) = 93° \, 20'.$

EXERCISES

1. Solve, by using the cosine formula, the triangles in problems (i) to (iv).

(i) $a = \sqrt{6}, b = 2, c = \sqrt{3} + 1.$

(ii) $a = \sqrt{6}, b = 2, c = \sqrt{3} + 1.$

(iii) $a = 8, b = 9, c = 10.$

(iv) $a = 16, b = 20, c = 33.$

2. For what values of x can $x^2 - 1$, $2x + 1$, $x^2 + x + 1$ be the lengths of the sides of a triangle. If x has any one of these values, prove that the greatest angle of the triangle is 120°.

3. Solve, without using the cosine formulae and the half-angle formulae the triangles in the following problems:
   - (i) $a = 9, b = 5, c = 8$.
   - (ii) $a = 7, b = 9, c = 10$.
   - (iii) $a = 11, b = 16\ c = 20$.

4. Prepare a complete outline of the logarithmic solution of the triangle (oblique) in which a, b, c are given.

5. Solve the triangles in the following problems by using logarithms, and check the solution in each case:
   - (i) $a = 25, b = 26, c = 27$.
   - (ii) $a = 229, b = 181, c = 257$.
   - (iii) $a = 4584, b = 3624, c = 5140$.
   - (iv) $a = 18{\cdot}77, b = 25{\cdot}32, c = 29{\cdot}65$.

6. Check the solutions of problems 1(iii), 2(i), 2(ii) by constructing the triangles accurately.

6.6. Case II. Two Sides and the Included Angle Given (SAS)

To solve a triangle ABC when b, c, and A are given. In this case the solution always exists and is unique. If logarithms are not to be used (see examples 1 (i) to (iv), then we first determine a by the cosine formula

$$a^2 = b^2 + c^2 - 2\,bc \cos A,$$

and then determine B and C by the sine formulae

$$\frac{\sin B}{b} = \frac{\sin A}{a} \text{ i.e., } \sin B = \frac{b \sin A}{a},$$

and $$\frac{\sin C}{c} = \frac{\sin A}{a} \text{ i.e., } \sin C = \frac{c \sin A}{a}.$$

If logarithmic computation is desired (see example 6), then we use the formulae

$$\tan \tfrac{1}{2} (B - C) = \frac{b - c}{b + c} \tan \tfrac{1}{2} (B + C)$$

and

$$a = \frac{b \sin A}{\sin B}.$$

The first of these formulae is given in suitable form when $b > c$. If $b < c$, the equivalent form obtained by interchanging b and B with c and C respectively is used. If $b = c$, then $B = C = 90 - A/2$, and the first formula is nor needed. Also, then, $a = 2b \sin A/2$.

To check the solution we may use one or more of the following:

(i) find $A + B + C$;

(ii) find B from the formula $\sin B = \dfrac{b \sin C}{c}$;

(iii) find a from the formula $a = \dfrac{c \sin A}{\sin C}$.

Example 5. Solve the triangle ABC, given $b = \sqrt{3} + 1$, $c = 2$, $A = 60°$.

Solution. We shall first find a. By the cosine formula for a,

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cos A \\ &= (\sqrt{3} + 1)^2 + 2^2 - 2.\,(\sqrt{3} + 1).\,2.\,\tfrac{1}{2} \\ &= 6, \end{aligned}$$

so that $a = \sqrt{6}$.

We shall next find C. By the sine formula,

$$\frac{a}{\sin A} = \frac{c}{\sin C},$$

i.e., $$\sin C = \frac{c \sin A}{a} = \frac{2.\,\sqrt{\frac{3}{4}}}{\sqrt{6}} = \frac{1}{\sqrt{2}},$$

so that $C = 45°$ or $135°$.

Since the sum of the angles A and C cannot exceed 180°, therefore the value $C = 135°$ is not admissible.

Thus $C = 45°$.

Now we shall find the remaining angle B.

$$B = 180° - (A + C) = 180° - (60° + 45°) = 75°.$$

Hence $a = \sqrt{6}$, $B = 75°$, $C = 45°$.

Check. By the sine formula,

$$\begin{aligned} \sin B = \frac{b \sin C}{c} &= \frac{(\sqrt{3} + 1)}{2\sqrt{2}} \\ &= \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} + \frac{1}{2} \cdot \frac{1}{\sqrt{2}} \\ &= \sin 75°, \end{aligned}$$

so that $B = 75°$ which agrees with the value of B found above by another method.

Example 6. Solve the triangle ABC, given $b = 5$, $c = 3$, $A = 120°$.

Solution. We shall first find a. By the cosine formula for a,

$$\begin{aligned} a^2 &= b^2 + c^2 - 2\,bc \cos A \\ &= 5^2 + 3^2 - 2.5.4.\,(-1/2) \\ &= 49, \end{aligned}$$

so that $a = 7$.

We shall next find C. By the sine formula,

$$\sin C = \frac{c \sin A}{a}$$
$$= \frac{3\sqrt{3}}{14}$$
$$= \cdot 3712,$$

so that C = 22°, to the nearest degree.

We shall now find B. By the sine formula,

$$\sin B = \frac{b \sin A}{a}$$
$$= \frac{5\sqrt{3}}{14}$$
$$= \cdot 6186.$$

so that B = 38°, to the nearest degree.

Check. $A + B + C = 120° + 38° + 22° = 180°$.

Remarks 1. Since the sides are given to one significant digit and the angle is given to the nearest degree, therefore we have determined the angles to the nearest degree.

2. $\sin C = \cdot 3712$ actually gives $C = 22°$ or $158°$.

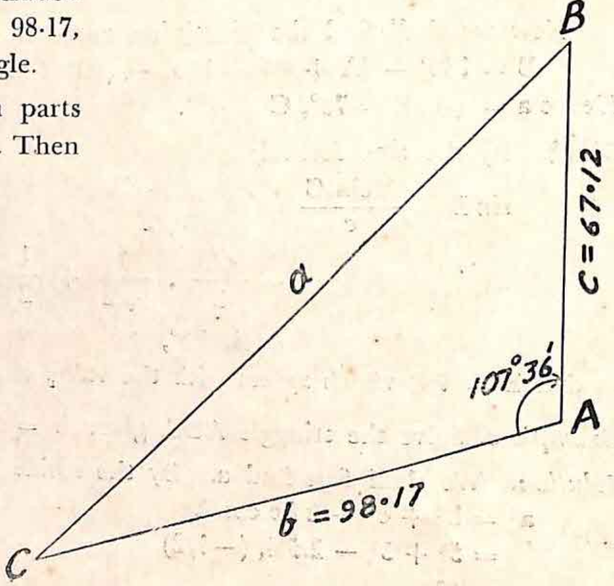
Since $A = 120°$, the value $C = 158°$ is obviously inadmissible. We therefore wrote $C = 22°$, without caring at all for the value 158°.

Example 7. Two sides and the included angle of a triangle have measures 98·17, 67·13 and 107° 36′. Solve the triangle.

Solution. Let us denote the given parts by b, c and A as shown in figure 6.4. Then

$$b = 98 \cdot 17$$
$$c = 67 \cdot 13$$

$$b - c = 31 \cdot 04$$
$$b + c = 165 \cdot 30$$
$$B + C = 180° - 107°\ 36'$$
$$= 70°\ 24'$$



Fig· 6.4

$$\tfrac{1}{2}(B + C) = 36^\circ\ 12' \rightarrow \tan\tfrac{1}{2}(B - C) = \frac{31{\cdot}04}{165{\cdot}30}\tan 36^\circ\ 12'.$$

$$\begin{array}{ll}
\log 31{\cdot}04 & = 1{\cdot}4920 \\
\log \tan 36^\circ\ 12' & = 9{\cdot}8644 - 10 \\
\hline
 & \text{A} \\
\log \text{num.} & = 11{\cdot}3564 - 10 \\
\log 165{\cdot}33 & = 2{\cdot}2183 \\
\hline
 & \text{S} \\
\log \tan\tfrac{1}{2}(B - C) & = 9{\cdot}1381 - 10
\end{array}$$

$$\begin{array}{l}
\underline{\tfrac{1}{2}(B - C) = 7^\circ\ 49'} \leftarrow \tfrac{1}{2}(B - C) = 7^\circ\ 49' \\
\quad B = 44^\circ\ 1' \\
\quad C = 28^\circ\ 23'.
\end{array}$$

To determine a, we have

$$a = \frac{b \sin A}{\sin B} = \frac{98{\cdot}17 \sin 107^\circ\ 36'}{\sin 44^\circ\ 1'}$$

$$\begin{array}{ll}
\log 98{\cdot}17 & = 1{\cdot}9920 \\
\log \sin 107^\circ\ 36' & = 9{\cdot}9804 - 10 \\
\hline
 & \text{A} \\
\log \text{num.} & = 11{\cdot}9724 - 10 \\
\log \sin 44^\circ\ 1' & = 9{\cdot}8419 - 10 \\
\hline
 & \text{S} \\
\log a & = 2{\cdot}1305 \\
a & = 135{\cdot}1
\end{array}$$

Check. To check the solution, we may determine a by using the formula

$$a = \frac{c \sin A}{\sin C}.$$

$$a = \frac{c \sin A}{\sin C} = \frac{67{\cdot}13 \sin 107^\circ\ 36'}{\sin 28^\circ\ 23'}$$

$$\begin{array}{ll}
\log 67{\cdot}13 & = 1{\cdot}8269 \\
\log \sin 28^\circ\ 23' & = 9{\cdot}9804 - 10 \\
\hline
 & \text{A} \\
\log \text{num.} & = 11{\cdot}8073 - 10 \\
\log \sin 28^\circ\ 23' & = 9{\cdot}6771 - 10 \\
\hline
 & \text{S} \\
\log a & = 2{\cdot}1302 \\
a & = 135{\cdot}0.
\end{array}$$

The check is fairly satisfactory. A difference of ·1 is not surprising.

EXERCISES

1. Solve, by using the cosine formula and the sine formulae, the triangles in the following problems. Check your solution in each case.

(i) $b = 1,\ c = \sqrt{3},\ A = 30°$.

(ii) $b = \sqrt{3} + 1,\ c = \sqrt{3} - 1,\ A = 60°$.

(iii) $a = 11,\ b = 20,\ C = 37°$.

(iv) $c = 23,\ a = 12,\ B = 35°$.

2. Prepare a complete outline of the logarithmic solution of the triangle in which the following parts are given :

(i) a, b, C with $a > b$

(ii) a, b, C with $b > a$.

(iii) b, c, A with $b > c$

(iv) b, c, A with $c > b$.

(v) c, a, B with $c > a$.

(vi) c, a, B with $a > c$.

3. Solve the triangles in the following problems by using logarithms. Check the solution in each case.

(i) $b = 540,\ c = 420,\ A = 52°\ 10'$.

(ii) $a = 35{\cdot}21,\ b = 21{\cdot}35,\ C = 50°\ 48'$.

(iii) $c = 164{\cdot}3,\ a = 242{\cdot}5,\ B = 54°\ 36'$.

(iv) $a = 158,\ b = 237,\ C = 66°\ 40'$.

4. The ratio of the two sides a and b of a triangle ABC is 7 : 3 and the angle included by these sides is 60°. Find, to the nearest 10′, the angles A and B.

5. Check the solutions of problems (iii) and (iv) by constructing the triangles to scale.

6.7. Case III. One Side and Two Angles Given

To solve the triangle ABC when a, B, C are given. The triangle exists if $B + C < 180°$ and is then unique.

The third angle is given by the formula

$$A = 180° - (B + C).$$

b and c are then determined by the sine formulae

$$b = \frac{a \sin B}{\sin A} \text{ and } c = \frac{a \sin C}{\sin A}$$

respectively.

Example 8. Solve the triangle ABC, given $a = 15,\ B = 37°,\ C = 43°$.

Solution :

(1) $A = 180° - (37° + 43°)$
$= 110°$.

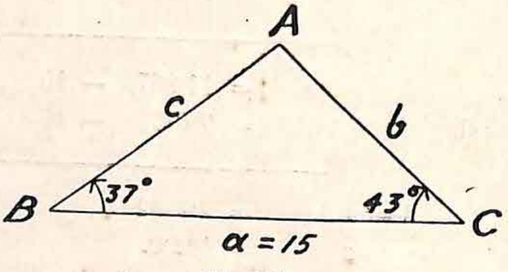


Fig. 6.5

(2) To find b, we have by using the sine formula

$$b = \frac{a \sin B}{\sin A} = \frac{15 \sin 37^\circ}{\sin 110^\circ} = \frac{15\,({\cdot}6018)}{{\cdot}9397} = 9{\cdot}6.$$

(3) To find c, we have by using the sine formula

$$c = \frac{a \sin C}{\sin A} = \frac{15 \sin 43^\circ}{\sin 110^\circ} = \frac{15\,({\cdot}6820)}{{\cdot}9397} = 11.$$

The computed parts are $A = 110^\circ$, $b = 9{\cdot}6$, $c = 11$. (The values of b and c have been rounded off to two-figure accuracy.)

Check. To check the results, let us find c by using the formula

$$c = \frac{b \sin C}{\sin B}.$$

$$c = \frac{9{\cdot}6 \sin 43^\circ}{\sin 37^\circ} = \frac{9{\cdot}6\,({\cdot}6820)}{{\cdot}6018} = 11,$$ when rounded off to two-figure accuracy.

Remark. We could also have used the cosine formula to check the result.

Example 9. Solve the triangle ABC, given $c = 48{\cdot}31$, $A = 31^\circ\, 29'$, $C = 70\,^\circ\, 43'$.

Solution: (1) $B = 180^\circ - (31^\circ\, 29' + 74^\circ\, 43')$
$= 78^\circ\, 48'$.

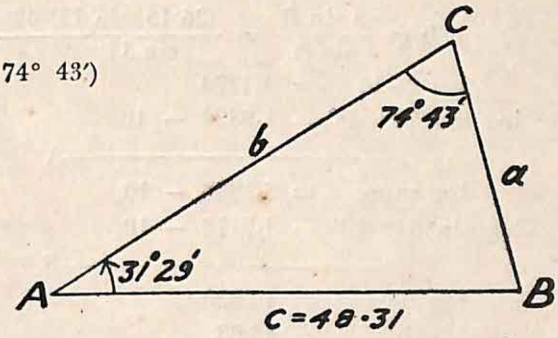


Fig. 6.6

$$\text{(2) } a = \frac{c \sin A}{\sin C} = \frac{48{\cdot}31 \sin 31^\circ\, 29'}{\sin 74^\circ\, 43'}.$$

On preparing an outline of the logarithmic solution and filling in the numbers, we have

$\log 48{\cdot}31 = 1{\cdot}6840$
$\log \sin 31^\circ\, 29' = 1{\cdot}7178 - 10$
A

$\log \text{num.} = 3{\cdot}4018 - 10$
$\log \sin 74^\circ\, 43' = 1{\cdot}9843 - 10$
S

$\log a = 1{\cdot}4175$
$a = 26{\cdot}15.$

$$(3)\ b = \frac{c \sin B}{\sin C} = \frac{48{\cdot}31 \sin 73^\circ 48'}{\sin 74^\circ 43'}.$$

On preparing an outline of the logarithmic solution and filling in the numbers, we have

$$\begin{array}{rll}
\log 48{\cdot}31 & = 1{\cdot}6840 & \\
\log \sin 73^\circ 48' & = 1{\cdot}9824 - 10 & \\
\hline
 & & \text{A} \\
\log \text{num.} & = 3{\cdot}6664 - 10 & \\
\log \sin 74^\circ 43' & = 1{\cdot}9843 - 10 & \\
\hline
 & & \text{S} \\
\log b & = 1{\cdot}6821 & \\
b & = 48{\cdot}09. &
\end{array}$$

The computed parts are B = 73° 48′, a = 26·15, b = 48·09.

Check. To check the results, let us find b by using the formula

$$b = \frac{a \sin B}{\sin A}.$$

$$b = \frac{a \sin B}{\sin A} = \frac{(26{\cdot}15) \sin 73^\circ 48'}{\sin 31^\circ 29'}.$$

$$\begin{array}{rll}
\log 26{\cdot}15 & = 1{\cdot}4174 & \\
\log \sin 73^\circ 48' & = 1{\cdot}9824 - 10 & \\
\hline
 & & \text{A} \\
\log \text{num.} & = 3{\cdot}3998 - 10 & \\
\log \sin 31^\circ 29' & = 1{\cdot}7178 - 10 & \\
\hline
 & & \text{S} \\
\log b & = 1{\cdot}6820 & \\
b & = 48{\cdot}08. &
\end{array}$$

The check is satisfactory. The difference of ·01 in the values of b obtained by two different methods is not surprising.

EXERCISES

1. Solve, without the use of logarithms, the triangles in the following problems. Check your solution in each case.

(i) B = 31°, C = 43°, a = 28.

(ii) A = 80°, B = 53°, b = 152.

(iii) C = 36° 20′, A = 45° 50′, c = 140.

(iv) A = 61° 10′, B = 36° 20′, a = 152.

2. Prepare an outline of the logarithmic solution of the triangle in which the following parts are given :

(i)	B, C, a	(iv)	C, A, a.
(ii)	B, C, b.	(v)	C, A, b.
(iii)	B, C, c.	(vi)	C, A, c.

3. Solve the triangles in the following problems by using logarithms. Check your solution in each case.

(i) $b = 39$, $A = 81°$, $B = 27°$.

(ii) $C = 136$, $B = 34° 20'$, $C = 67° 30'$.

(iii) $a = 15{\cdot}72$, $A = 41° 30'$, $B = 72° 45'$.

(iv) $b = 28{\cdot}94$, $C = 61° 48'$, $B = 84° 23'$.

4. Check the solutions of problems 3(ii) and 3(iii) by using the formula

$$\frac{a+b}{c} = \frac{\cos \frac{1}{2}(A-B)}{\sin \frac{1}{2} C}.$$

5. Check the solution of problem 3(iv) by using the formula

$$\frac{a-b}{c} = \frac{\sin \frac{1}{2}(A-B).}{\cos \frac{1}{2} C}.$$

6.8. Case IV. Two Sides and One Angle Given

To solve the triangle ABC when a, b, A are given. There may exist no, one or two triangles depending on the relation between the given parts as we shall see below. Because of the possibility of having two triangles, this case is called the *Ambiguous Case.*

To discuss the existence and uniqueness of the solution we shall proceed geometrically.

We construct angle A and cut off AC = b. This fixes the vertex C. With C as centre we draw an arc in order to locate (if possible) B.

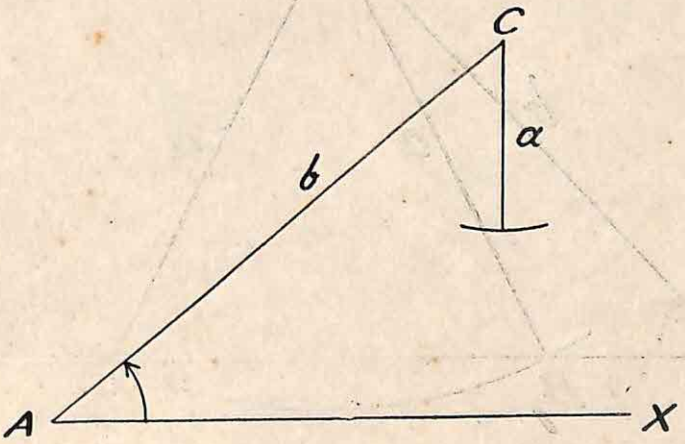


Fig. 6.7

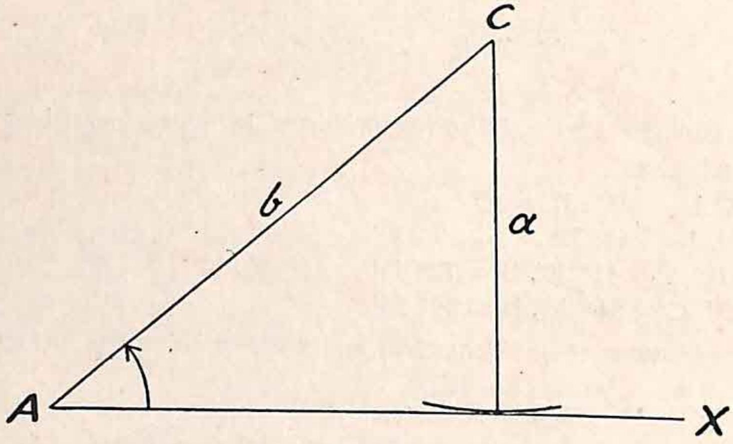


Fig. 6.8

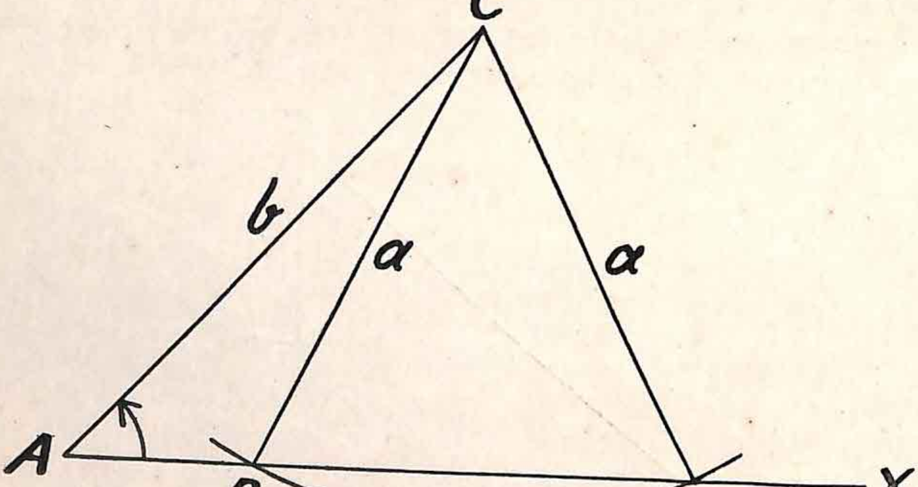


Fig. 6.9

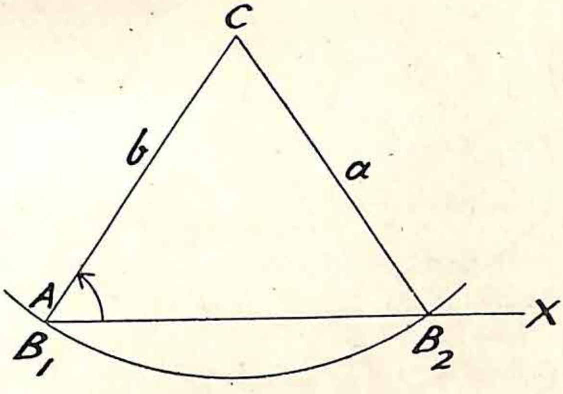


Fig. 6.10

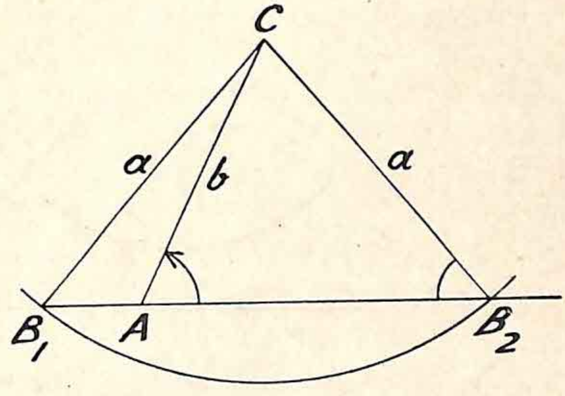


Fig. 6.11

For the sake of convenience let us consider the cases $A < 90°$, $A > 90°$ separately.

(α) $A < 90°$: Several possibilities arise :

(i) If $a < p$ (where $p = b \sin A$) is the perpendicular from C on AX, then the arc does not cut AX and no triangle is possible (Fig. 6.7).

(ii) If $a = p$, then the arc touches AX. Therefore one triangle is possible and it is right-angled (Fig. 6.8).

(iii) If $a > p$, then the arc cuts AX at two points. Both these points lie to the right of A if $a < b$ (Fig. 6.9), one of them lies to the right of A and the other coincides with A if $a = b$, (Fig. 6.10), and one of them lies to the right of A and the other to the left of A if $a > b$ (Fig. 6.11). Thus two triangles are possible if $a < b$ and only one triangle is possible if $a \geqslant b$. Because of the possibility of two triangles, the case $a < b$, $a > b \sin A$, A acute is called the *Ambiguous Case.*

(β) $A > 90°$: The following possibilities arise :

(i) If $a \leqslant b$ the arc does not cut AX at any point to the right of A and no triangle is possible (Fig. 6.12(a) and Fig. 6.12(b)).

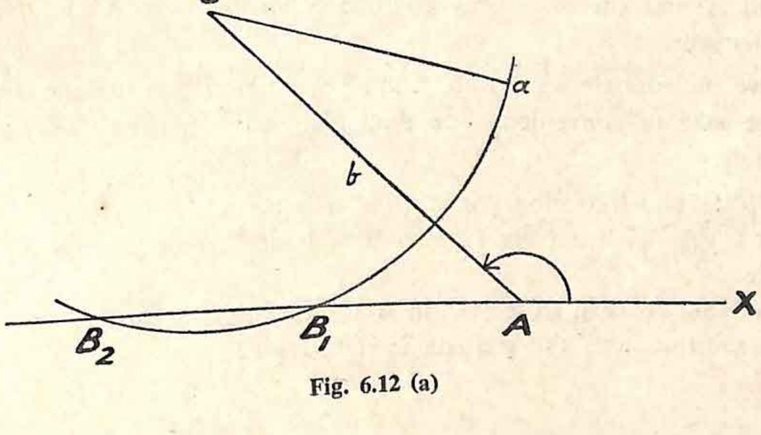


Fig. 6.12 (a)

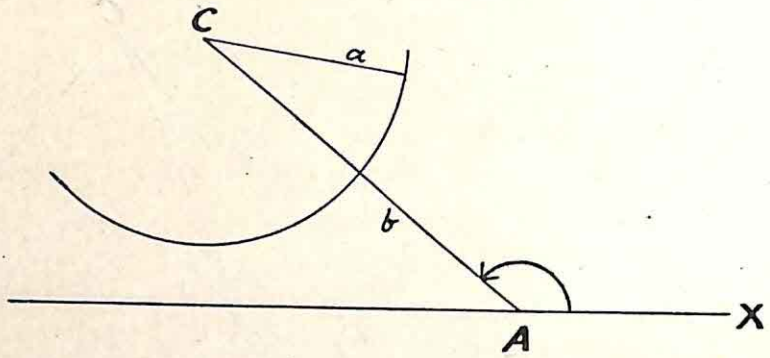


Fig. 6.12 (b)

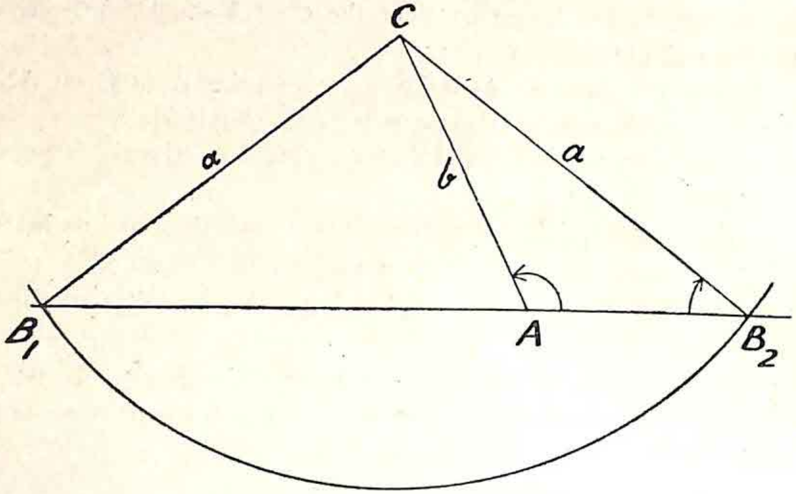


Fig. 6.13

(ii) If $a > b$, then the arc cuts AX at two points, only one of which lies to the right of A and therefore only triangle is possible (Fig. 6.13). This completes the discussion.

To solve the triangle when a, b, A are given we have to use the sine formula. For the sake of convenience we shall discuss the case $A < 90°$, and $A > 90°$ separately.

(α) $A < 90°$: The following possibilities arise :

(i) If $a < b \sin A$, then from (2), $\sin B > 1$ and consequently no solution is possible.

(ii) If $a = b \sin A$, then from (2), $\sin B = 1$, so that $B = 90°$. Therefore there is one solution and the triangle is right-angled.

(iii) If $a > b \sin A$, then (2) gives two values of B, one of which is acute and the other obtuse.

If $a \geqslant b$, then $A \geqslant B$, so that only the acute value of B is permissible, and consequently there is only one solution.

If $a < b$, then $A < B$, so that both the values of B are possible, and consequently there may be two solutions.

(β) $A > 90°$: The following possibilities arise :

(i) If $a \leqslant b$, then $A \leqslant B$, so that B must also be an obtuse angle, which is impossible. Hence no solution is possible.

(ii) If $a > b$, then only the acute value of B is permissible and therefore only one triangle is possible.

Having determined B (whenever there exists a permissible value of B), we determine C by the formula $C = 180° - (A + B)$. The remaining side c is then found as in the SAS case. In the ambiguous case the values of C and c corresponding to the two values of B have to be found separately.

If we use logarithms, then it may be helpful to observe that

$\log \sin B > 0 \Rightarrow \sin B > 1 \Rightarrow$ no solution.

$\log \sin B = 0 \Rightarrow \sin B = 1 \Rightarrow B = 90°$.

$\log \sin B < 0 \Rightarrow \sin B < 1$ i.e., there are two values of B say B_1 and B_2

We then consider these values as discussed above.

Remark. The above case can be disposed of by using the cosine formula also.

If a, b, A are given, then the cosine formula for a gives

$$a^2 = b^2 + c^2 - 2bc \cos A,$$

$$\text{or } c^2 - 2bc \cos A + b^2 - a^2 = 0. \qquad \text{...........(i)}$$

Solving (i) as a quadratic in c, we have

$$c = \frac{2\, b \cos A \pm \sqrt{\{4\, b^2 \cos^2 A - 4\,(b^2 - a^2)\}}}{2}$$

$$= b \cos A \pm \sqrt{(a^2 - b^2 \sin^2 A)}. \qquad \text{...........(ii)}$$

Since c is the length of a side of a triangle, therefore it must be positive. We have therefore to determine as to how many of the values of c given by (ii) are positive for any given set of values of a, b and A.

Two different possibilities arise :

(i) $A < 90°$: If $A < 90°$, $\cos A$ is positive so that $b \cos A$ is positive.

Three sub-cases arise :

(α) If $a < b \sin A$, then $a^2 < b^2 \sin^2 A$, so that $a^2 - b^2 \sin^2 A < 0$. The two values of c are imaginary and no triangle is possible.

(β) If $a = b \sin A$, then $a^2 = b^2 \sin^2 A$, so that $a^2 - b^2 \sin^2 A = 0$. There is only one value of c ($= b \cos A$) from (ii) which is positive, therefore only one traingle is possible.

(γ) If $a > b \sin A$, then $a^2 > b^2 \sin^2 A$ so that $a^2 - b^2 \sin^2 A > 0$. In this case (ii) gives two real and distinct values of c. One of these values, namely

$$b \cos A + \sqrt{\{a^2 - b^2 \sin^2 A\}}$$

is surely positive ; the other value

$$b \cos A - \sqrt{\{a^2 - b^2 \sin^2 A\}}$$

is positive if

$$b \cos A > \sqrt{\{a^2 - b^2 \sin^2 A\}}$$

i.e., if $b^2 \cos^2 A > a^2 - b^2 \sin^2 A$,

i.e., if $b^2 > a^2$

i.e., if $b > a$.

Therefore two triangles are possible if $b > a$ and only one triangle is possible if $b \leqslant a$.

(ii) $A > 90°$: If $A > 90°$, cos A is negative so that b cos A is negative.

The value $b \cos A \sqrt{\{a^2 - b^2 \sin^2 A\}}$ is surely negative.

The value $b \cos A + \sqrt{\{a^2 - b^2 \sin^2 A\}}$ is positive if

$$\sqrt{\{a^2 - b^2 \sin^2 A\}} > -b \cos A,$$

i.e., if $a^2 - b^2 \sin^2 A > b^2 \cos^2 A$,

i.e., if $a^2 > b^2$

i.e., if $a > b$.

Thus we find that when $A > 90°$, no triangle is possible when $a \leqslant b$ and only one triangle is possible when $a > b$.

SUMMARY OF THE VARIOUS POSSIBILITIES

(i) $A < 90°$

(α) $a < b \sin A$	No triangle
(β) $a = b \sin A$	One triangle
(γ) $a > b \sin A$	Two triangles if $a < b$; one triangle if $a \leqslant b$.

(ii) $A > 90°$

(α) $a \leqslant b$	No triangle
(β) $a > b$	One triangle.

Example 10. Given $a = 19$, $b = 31$, $A = 56°$, either show that there is no solution or find all solutions.

Solution. We shall use the sine formula to find B.

$$\sin B = \frac{b \sin A}{a} = \frac{31 \sin 56°}{19} = \frac{31\ (\cdot 8290)}{19} = 1\cdot 3525.$$

Since sin B > 1, there does not exist any solution.

Remark. The above example illustrates the case $A < 90°$, $a < b \sin A$.

Example 11. Given $a = 31{\cdot}1$, $b = 62{\cdot}2$, $A = 30°$, either show that there is no solution or find all solutions.

Solution (1) We shall use the sine formula to find B.

$$\sin B = \frac{b \sin A}{a} = \frac{(62\cdot 2)\ \frac{1}{2}}{31\cdot 1} = 1.$$

Since sin B = 1, therefore B = 90°. There exists only one solution.

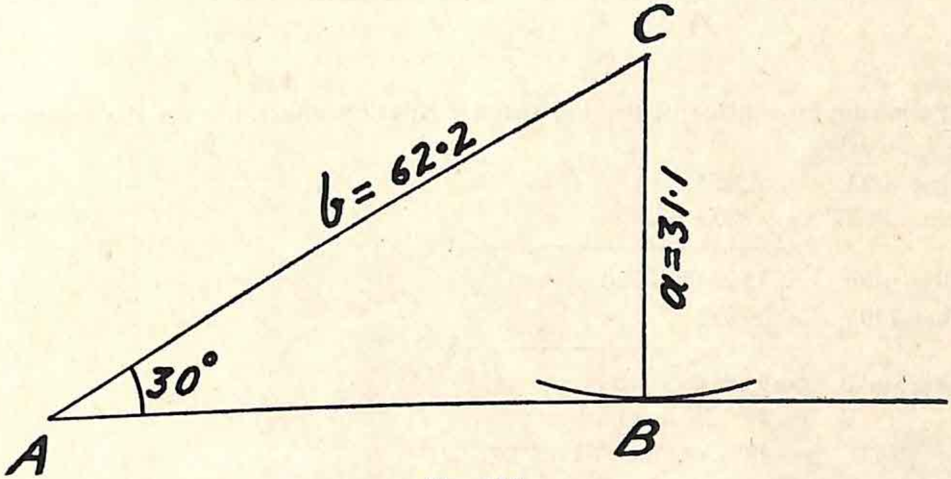


Fig. 6.14

(2) $C = 180° - (30° + 90°) = 60°$.

(3) To find c, we shall use the sine formula

$$c = \frac{b \sin C}{\sin B} = 62\cdot 2 \sin 60° = (62\cdot 2).\ (\cdot 8660) = 53\cdot 9,$$

when rounded off to three-figure accuracy.

Remark. The above example illustrates the case $A < 90°$, $a = b \sin A$.

Example 12. Given $a = 7295$, $b = 8231$, $A = 42°\ 27'$, either show that there is no solution or find all solutions.

Solution. (1) We shall use the sine formula to find B.

$$\sin B = \frac{b \sin A}{a} = \frac{82\cdot 31\ (\sin 42°\ 27')}{7295}.$$

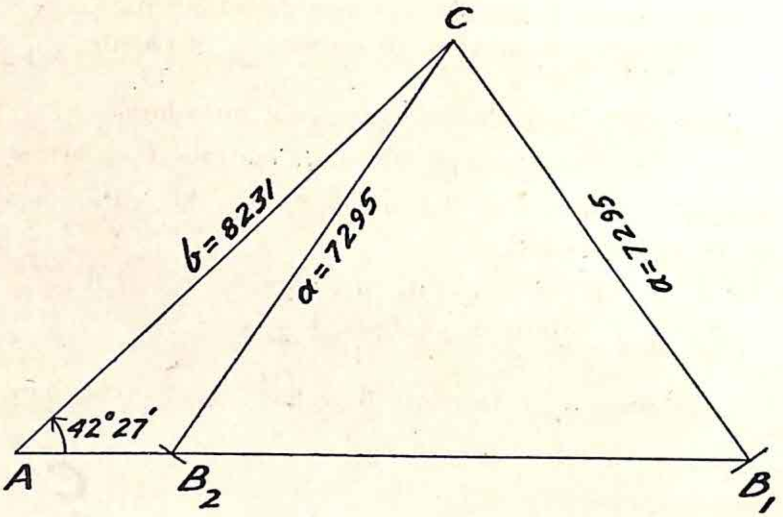


Fig. 6.15

Preparing an outline of the logarithmic solution and filling in the numbers we have

$\log 8231 = 3{\cdot}9155$

$\log \sin 42^\circ\, 27' = 9{\cdot}8293 - 10$

———————————— A

$\log \text{num.} = 13{\cdot}7448 - 10$

$\log 7295 = 3{\cdot}8632$

———————————— S

$\log \sin B = 9{\cdot}8816 - 10$

$B = 49^\circ\, 36'$ or $130^\circ\, 24'$.

Let $B_1 = 49^\circ\, 36'$, $B_2 = 130^\circ\, 24'$.

Since $A + B_2 < 180^\circ$, therefore both the values are admissible and there are two solutions.

(2) $C_1 = 180^\circ - (A + B_1)$

$= 180^\circ - (42^\circ 27' + 49^\circ 36')$

$= 87^\circ\, 57'$.

$C_2 = 180^\circ - (A + B_1)$

$= 180^\circ - (42^\circ\, 27' + 130^\circ\, 24')$

$= 7^\circ\, 9'$.

(3) $$C_1 = \frac{a \sin C_1}{\text{Sin } A} = \frac{7295 \sin 7^\circ\, 57'}{\sin 42^\circ\, 27'}$$

$\log 7295 = 3.8632$

$\log \sin 87^\circ\, 57' = 9{\cdot}9997 - 10$

———————————— A

$\log \text{num.} = 13{\cdot}8629 - 10$

$$c_2 = \frac{a \sin C_2}{\sin A} = \frac{7295 \sin 7^\circ\, 9'}{\sin 42^\circ\, 27'}$$

$\log 7295 = 3{\cdot}8632$

$\log \sin 7^\circ\, 9' = 9{\cdot}0949 - 10$

———————————— A

$\log \text{num.} = 12{\cdot}9581 - 10$

$$\begin{aligned} \log \sin 42^\circ\, 27' &= 9{\cdot}8293 - 10 \\ &\underline{\qquad\qquad\qquad} S \\ \log c_1 &= 4{\cdot}0336 \\ c_1 &= 10800. \end{aligned} \qquad\qquad \begin{aligned} \log \sin 42^\circ\, 27_1 &= 9{\cdot}8293 - 10 \\ &\underline{\qquad\qquad\qquad} S \\ \log c_2 &= 3{\cdot}1288 \\ c_2 &= 1345. \end{aligned}$$

Thus there exist two solutions. The computed parts are

$$B_1 = 49^\circ\, 36',\ C_1 = 87^\circ\, 57',\ c_1 = 10800\,;$$
$$B_2 = 130^\circ\, 24',\ C_2 = 7^\circ\, 9',\ c_2 = 1345.$$

Remark. The above example illustrates the ambiguous case $A < 90^\circ$, a b sin A and $a < b$. The solutions can be checked by using the formula $c = \frac{b \sin C}{\sin B}$ for determining the values of c.

Example 13. Given $a = 531{\cdot}4$, $b = 424{\cdot}9$, $A = 70^\circ\ 15'$, either show that there is no solution or find all the solutions.

Solution (1) We shall use the sine formula to find B.

$$\sin\ B = \frac{b \sin A}{a} = \frac{421{\cdot}9 \sin 70^\circ\ 15'}{531{\cdot}4}.$$

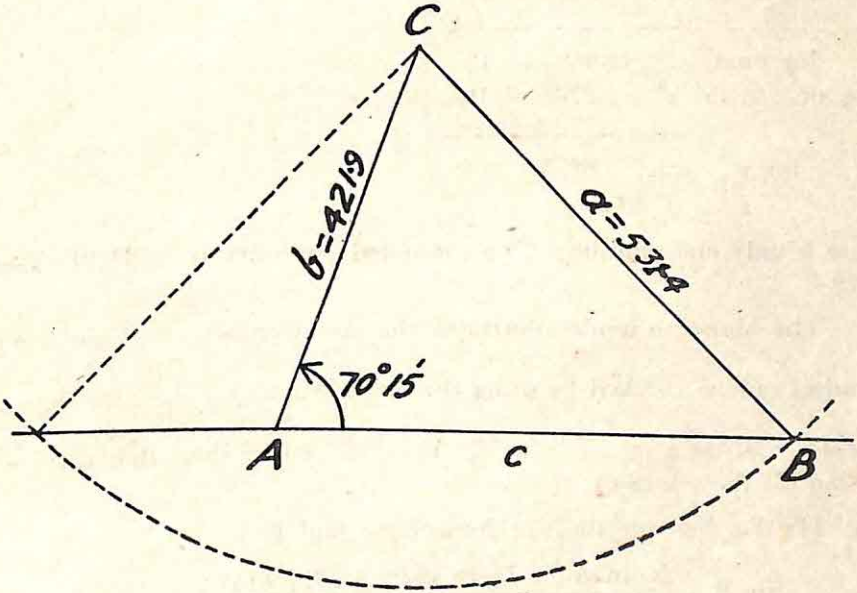


Fig. 6.16

Preparing an outline of the logarithmic solution and filling in the numbers we have

$$\begin{aligned} \log 421{\cdot}9 &= 2{\cdot}6252 \\ \log \sin 70^\circ\ 15' &= 9{\cdot}9736 - 10 \\ &\underline{\qquad\qquad\qquad} A \end{aligned}$$

$$\begin{array}{rll}
\log \text{num.} & = 12{\cdot}5988 - 10 & \\
\log 531{\cdot}4 & = 2{\cdot}7254 & \\
\cline{2-2}
 & & S \\
\log \sin B & = 9{\cdot}8734 - 10 & \\
B & = 48^\circ\ 21' \text{ or } 131^\circ\ 39'. &
\end{array}$$

Since $131^\circ\ 39' + 70^\circ\ 15'$ exceeds 180°, therefore the value $131^\circ\ 38'$ is inadmissible, so that $B = 48^\circ\ 21'$.

Therefore there is only one solution.

(2) $C = 180^\circ - (A + B) = 180^\circ - (70^\circ\ 15' + 48^\circ\ 21') = 61^\circ\ 24'$

(3) We shall use the sine formula to determine c

$$c = \frac{a \sin C}{\sin A} = \frac{531{\cdot}4 \sin 61^\circ\ 24'}{\sin 70^\circ\ 15'}\ .$$

Preparing an outline of the logarithmic solution and filling in the numbers we have

$$\begin{array}{rll}
\log 531{\cdot}4 & = 2{\cdot}7254 & \\
\log \sin 61^\circ\ 24' & = 9{\cdot}9435 - 10 & \\
\cline{2-2}
 & & A \\
\log \text{num.} & = 12{\cdot}6689 - 10 & \\
\log \sin 70^\circ\ 15' & = 9{\cdot}9736 - 10 & \\
\cline{2-2}
 & & S \\
\log c & = 2{\cdot}6953 & \\
c & = 495{\cdot}8. &
\end{array}$$

There is only one solution. The computed parts are $B = 48^\circ\ 21'$, $C = 61^\circ\ 24'$, $c = 495{\cdot}8$.

Remark. The above example illustrates the case $A < 90^\circ$, $a > b \sin A$, and $a > b$.

The solution can be checked by using the formula $c = \dfrac{b \sin C}{\sin B}$ for determining c.

Example 14. Given $a = 24$, $b = 27$, $A = 142^\circ$, either show that there is no solution or find all the solutions.

Solution. (1) We shall use the sine formula to find B.

$$\sin B = \frac{b \sin A}{a} = \frac{27 \sin 142^\circ}{24} = \frac{27.\,(\cdot 6157)}{24} = \cdot 6927,$$

so that $B = 44^\circ$ or 136°, to the nearest degree.

For each of these values of B we find that $A + B > 180^\circ$. Therefore none of the values of B is admissible. Hence there is no solution.

Remark. The above example illustrates the case $A > 90^\circ$, $a < b$.

Example 15. Given $a = 27{\cdot}3$, $b = 19{\cdot}1$, $A = 108^\circ 20'$, show that there is only one solution and find B.

Solution. By the sine formula we have

$$\sin B = \frac{b \sin A}{a} = \frac{19{\cdot}1 \sin 108^\circ 20'}{27{\cdot}3}.$$

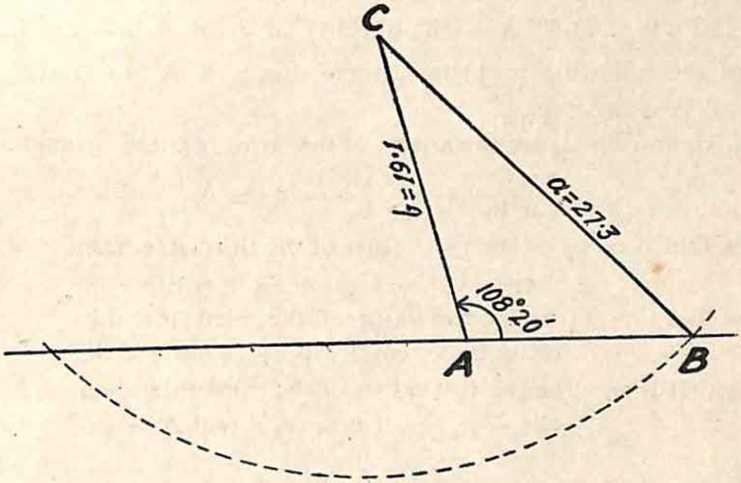


Fig. 6.17

Preparing an outline of the logarithmic solution and filling in the numbers we have

$$\begin{array}{rll} \log 19{\cdot}1 = & 1{\cdot}2810 & \\ \log \sin 108^\circ 20' = & 9{\cdot}9774 - 10 & \\ \hline & & \text{A} \\ \log \text{num.} = & 11{\cdot}2584 - 10 & \\ \log 27{\cdot}3 = & 1{\cdot}4362 & \\ \hline & & \text{S} \\ \log \sin B = & 9{\cdot}8222 - 10 & \end{array}$$

$\therefore B = 41^\circ 37'$ or $138^\circ 23'$.

Since $A > 90^\circ$, the second one of the two values of B is obviously inadmissible. Also, since $41^\circ 37' + 108^\circ 20' < 180^\circ$, therefore $41^\circ 37'$ is an admissible value of B.

Hence there is only one solution and $B = 41^\circ 37'$.

Remark. The above example illustrates the case $A > 90^\circ$, $a > b$.

EXERCISES

1. In each of the following problems, either show that there is no solution or find all the solutions. Do not use logarithms in problems (i) to (vi).

(i) $a = 28,\ b = 17,\ A = 108^\circ$. (ii) $a = 32,\ b = 25,\ A = 29^\circ$.
(iii) $a = 69,\ b = 49,\ B = 37^\circ$. (iv) $a = 24,\ b = 31,\ A = 128^\circ$.
(v) $a = 82,\ b = 130,\ A = 44^\circ$. (vi) $a = 44,\ b = 71,\ A = 32^\circ$.
(vii) $a = 752,\ b = 84{\cdot}8,\ A = 65^\circ\ 30'$. (viii) $a = 32{\cdot}2,\ b = 25{\cdot}1,\ A = 51^\circ\ 10'$.
(ix) $a = 182,\ b = 244,\ A = 43^\circ\ 30'$. (x) $a = 23{\cdot}6,\ b = 18{\cdot}5,\ A = 101^\circ\ 20'$.
(xi) $a = 327,\ b = 537,\ A = 33^\circ\ 40'$. (xii) $b = 53{\cdot}4,\ c = 42{\cdot}6,\ C = 111^\circ\ 10'$.
(xiii) $a = 43{\cdot}75,\ b = 71{\cdot}38,\ A = 38^\circ\ 8'$. (xiv) $a = 29{\cdot}36,\ b = 25{\cdot}74,\ A = 52^\circ\ 12'$.

2. In each of the following problems assume that a, b, A are given, A is an acute angle and $b \sin A < a < b$.

(i) If B_1, C_1 and B_2, C_2 are the angles of the two possible triangles, prove that

$$\frac{\sin C_1}{\sin B_1} + \frac{\sin C_2}{\sin B_2} = 2 \cos A.$$

(ii) Show that if c_1, c_2 be the two values of the third side, then

$$| c_1 - c_2 | = 2\sqrt{\{ a^2 - b^2 \sin^2 A \}}.$$

(iii) Show that if c_1, c_2 be the two values of the third side, then

$$c_1^2 - 2c_1 c_2 \cos 2A + c_2^2 = 4a^2 \cos^2 A.$$

(iv) Show that if c_1, c_2 be the two values of the third side, then

$$(c_1 - c_2)^2 + (c_1 + c_2)^2 \tan^2 A = 4a^2.$$

6.9. Summary

The following table gives a summary of the formulae to be used for solving triangles in the various cases discussed above.

Given parts	Formulae to be used	
	without use of logarithms	with use of logarithms
SSS	Cosine formulae	Half-angle formulae
SAS	Cosine formulae Sine formulae	Napier's formulae Sine formulae
SSA	Sine formulae	Sine formulae
SAA	Sine formulae	Sine formulae

A problem on solution of triangles should be taken as a challenge and not merely as a routine problem where the rules set out in the book are to be applied. An ingenuity on the part of the reader may often reduce the labour considerably. For example, (i) in the SSS case, the triangle may be solved without the use of any special formulae (see example 4, page 151); (ii) one may compute one part without the aid of logarithms and another part with the aid of logarithms; (iii) when four parts are known, one may like to use the sine formula in the SSS or SAS case.

6.10. Applications

Solution of triangles has enormous applications to surveying, navigation, etc. We shall now consider some simple ones from among them. For this purpose, we need to explain certain terms that are generally used in practical problems.

Let O denote the position of the observer's eye and let P denote a certain point on some object which the observer is watching. The straight line from O to P is called the observer's *line of sight*. *Let* O H be the horizontal line through O lying in the vertical plane containing OP. Then, if OP be above OH, the angle HOP is called the *angle of elevation* of P at O. If OP be below OH, the angle HOP is called the *angle of depression* of P at O.

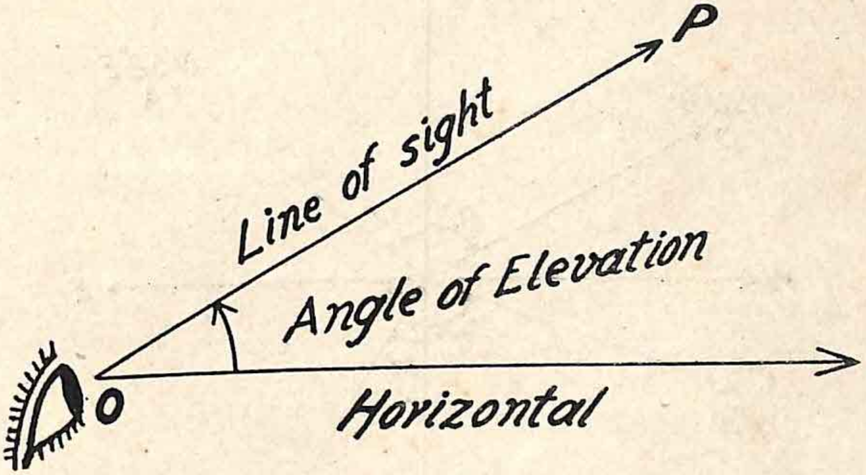


Fig. 6.18

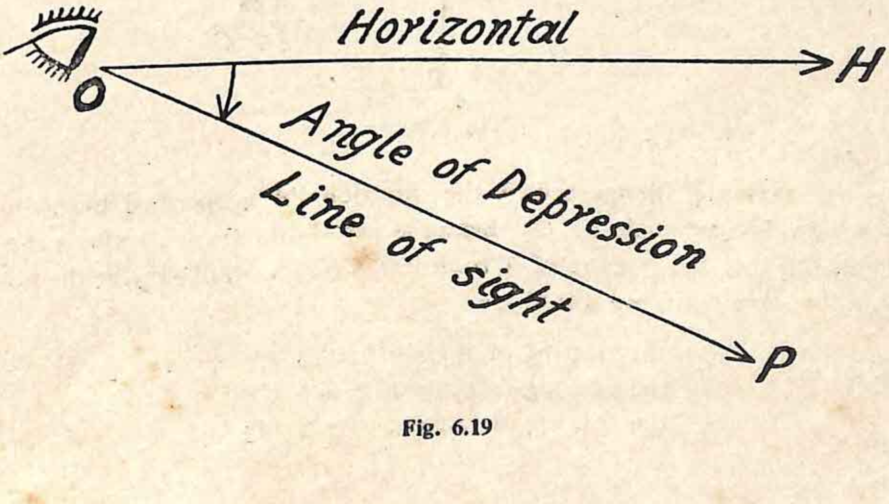


Fig. 6.19

Surveyors indicate the direction from one point to another in a rather special way. To understand it, let us denote by NS a horizontal line in the north-south direction. Let P be any point on NS and let WPE be the horizontal line through P perpendicular to NE. Then WPE is the west-east line through P.

The direction from P to any point in the horizontal plane through P can be specified if we know (i) the positive angle which the line makes with NS (ii) whether the angle is measured from PN or from PS (iii) whether the angle is measured to the west or the east.

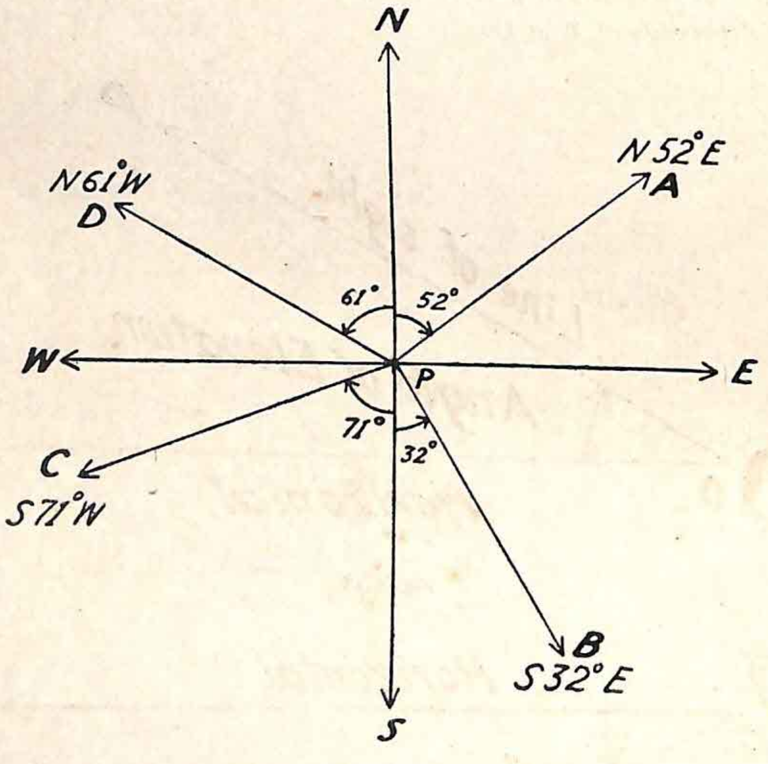


Fig. 6.20

Thus, for example, in figure 6·20, the direction PA is specified by saying that PA makes with PN an angle of 52° measured east-wards. We can express this more simply by saying that the *bearing* of A from P is N 52° E. Similarly, for the points B, C and D in the same figure we would say :

the bearing of B from P is S 32° E;
the bearing of C from P is S 71° W;
the bearing of D from P is N 61° W.

6.11. Problems Involving Right-Angled Triangles

Example 16. A balloon is directly above one end of a bridge 850 m. long. The angle of depression of the other end of the bridge from the balloon is 58°. Find the height of balloon above the bridge.

Solution. In the figure below, let AC indicate the bridge, B the balloon, and B H the horizontal through B lying in the plane B A C. Then $\angle HBC = 58°$. In the right-angled triangle BAC, AC = 850 m., $\angle ACB = \angle CBH = 58°$. The height h is given by

$$\frac{h}{850} = \tan 58°,$$

$$\text{so that } h = 850 \tan 58° = 850 \times 1.600 = 1360.$$

Hence the required height is 1360 metres.

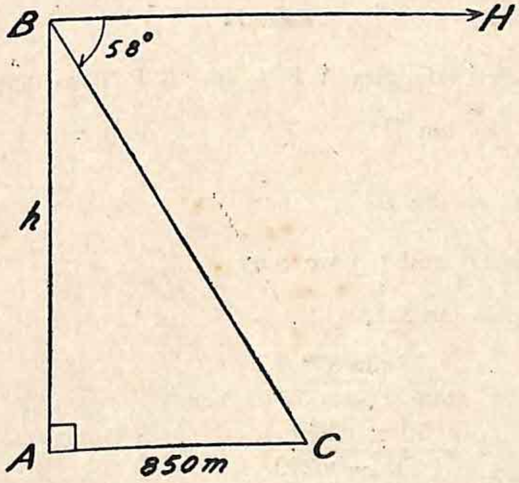


Fig. 6.21

Example 17. The angular elevation of the top of a vertical tower from a point A in the same horizontal plane as the foot of the tower is 51° and from a point B in the same horizontal line with the foot of the tower as A and 73 metres further away from it is 32°. Find the height of the tower and the distance of A from the foot of the tower.

Solution. In the figure below, let F and T denote the foot and the top (respectively) of the tower. Also let h metres be the height of the tower, and x metres the distance of A from the foot of the tower.

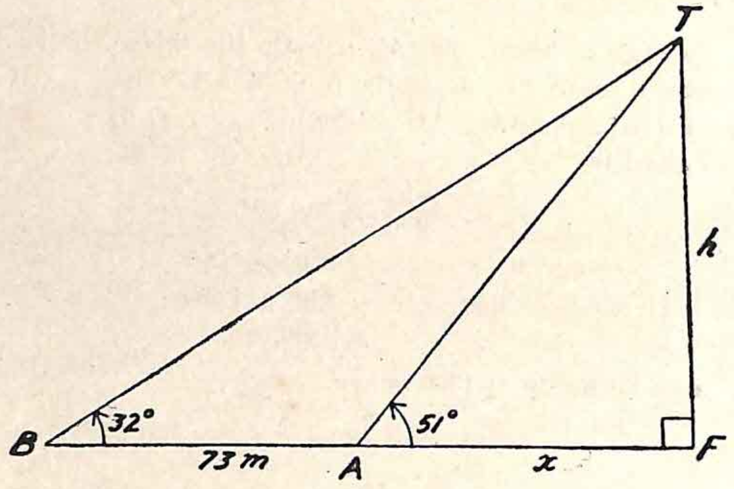


Fig. 6.22

From the right-angled triangles A F T and B F T we have

$$\frac{h}{x} = \tan 51° \qquad \text{............(i)}$$

$$\frac{h}{x+73} = \tan 32° \qquad \text{............(ii)}$$

Eliminating h from (i) and (ii) we have

$$\frac{x \tan 51°}{x + 73} = \tan 32°,$$

$$\text{or} \quad x = \frac{73 \tan 32°}{\tan 51° - \tan 32°}$$

$$= \frac{73 \times \cdot 6249}{1 \cdot 235 - \cdot 6249}$$

$$= \frac{73 \times \cdot 6249}{\cdot 5801}$$

$$= \frac{73 \times 6249}{5801},$$

so that

$$\log x = \log 73 + \log 6249 - \log 5801$$

$$= 1 \cdot 8633 + 3 \cdot 7958 - 3 \cdot 7635$$

$$= 1 \cdot 8956,$$

$$
\begin{aligned}
\text{i.e.,}\qquad x &= \text{antilog } 1{\cdot}8956 \\
&= 78{\cdot}63. \\
\text{Also,}\qquad h &= x \tan 51^\circ \\
&= 78{\cdot}63 \times 1{\cdot}235, \\
\text{so that } \log h &= \log 78{\cdot}63 + \log 1{\cdot}235 \\
&= 1{\cdot}8956 + {\cdot}0917 \\
&= 1{\cdot}9873, \\
\text{whence}\qquad h &= 97{\cdot}12.
\end{aligned}
$$

Hence the height of the tower is 97 metres and the distance of A from the foot of the tower is 79 metres (approx.)

Example 18. A statue x metres high standing on the top of a column y metres high, on the bank of a river, subtends at a point on the opposite bank directly facing the column, the same angle as that subtended at the same point by a man z metres high at the base of column. Find the breadth of the river.

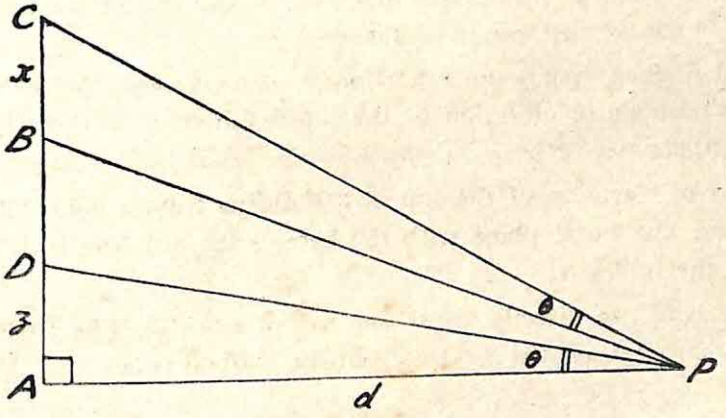


Fig. 6.23

Solution. In figure 6·23, let AB represent the column, BC represent the statue, and AD represent the man standing at the base of the column. Let P be the point on the opposite bank of the river, directly facing A. Let d = AP be the breadth of the river, $\theta = \angle$ APD $= \angle$ BPC be the angle which the statue subtends at P, and let $\angle$ DPB $= \phi$.

$$\text{Then } \tan\theta = \frac{AD}{AP} = \frac{z}{d}. \qquad \ldots\ldots(1)$$

$$\tan(\theta + \phi) = \frac{AB}{AP} = \frac{y}{d}. \qquad \ldots\ldots(2)$$

$$\tan(2\theta + \phi) = \frac{AC}{AP} = \frac{x + y}{d}. \qquad \ldots\ldots(3)$$

From (2) and (3) we find that

$$\tan \theta = \frac{(x + y)/d - y/d}{1 + (x + y)/d \cdot y/d}$$

$$= \frac{d\,x}{d^2 + y\,(x + y)}. \qquad \text{.........(4)}$$

From (1) and (4), we have

$$\frac{z}{d} = \frac{d\,x}{d^2 + y\,(x + y)},$$

i.e., $d^2\,(x - z) = (x + y)\,y\,z,$

whence $d = \sqrt{\{(x + y)\,y\,z/(x - z)\}}.$

EXERCISES

1. A ladder 5 metres long rests against a vertical wall, so that the angle between the ladder and the ground is 26°. Find the distance of the top of the ladder from the ground.

2. A man on a cliff-top 100 metres high sees a ship out at sea 425 metres away. Find the angle of depression of the ship.

3. A vertical flagstaff stands on a horizontal plane. From a point 50 metres from its foot, the angle of elevation of the top is found to be 30°. Find the height of the flagstaff.

4. The angle of elevation of the top of a building from a point 225 m from and in the same horizontal plane with the base of the building is 21° 10′. Find out height of the building.

5. A balloon is 675 m directly above one end of a bridge. The angle of depression of the other end of the bridge from the balloon is 56° 40′. How long is the bridge ?

6. From a ship, the angle of elevation of a point A at the top of a cliff is 19°. After the ship has sailed 800 m directly towards the foot of the cliff, the angle of elevation of A is 44°. Find the height of the cliff.

7. The angular elevation of the top of a vertical tower from a point A in the same horizontal plane as the foot of the tower is 60° and from a point B in the same horizontal line with the foot of the tower as A and 40 m further away is 30°. Find the height of the tower and also the distance of A from the foot of the tower.

8. AB is a straight horizontal road leading to F, the foot of a vertical tower, A being at a distance of 160 m from C, and B 100 m nearer. If the angle of elevation of the top of the tower at B be double the angle of elevation at A, find the height of the tower.

9. From a point A in the same horizontal plane as the foot of a vertical tower, the angles of elevation of two points B and C on the tower are 60° and 30° respectively. If BC = 40 m, find the height of B from the foot of the tower, and the distance of A from the foot of the tower.

10. Two vertical pillars of equal height stand on either side of a roadway which is 50 m wide. At a point in the roadway between the pillars the elevations of the tops of the pillars are 60° and 30°. Find the height of each pillar and the position of the point.

11. A ladder 8 m long reaches to a distance 8 m from the top of a vertical flagstaff. At the foot of the ladder the elevation of the top of the flagstaff is 60°. Find the height of the flagstaff.

12. A man on the top of a vertical light-house observes a boat coming directly towards it. If it takes 10 minutes for the angle of depression to change from 30° to 60°, how soon will it reach the light-house ?

13. A statue 6·80 m high standing on the top of a column 54·40 m high, on the bank of a river, subtends at a point on the opposite bank directly facing the column, the same angle as that subtended at the same point by a man 170 cm high at the base of the column. Find the breadth of the river.

14. A 9-metre flag-pole stands on the top of a 4·5 metres high building. How far from the base of the building should a man stand if the flag-pole and the building are to subtend equal angles at his eye which is 1·5 metres above the ground.

15. A vertical telegraph pole is supported by two wires, each running from the top of the pole to the ground. One wire is 23 m long and makes an angle of 52° with the ground. Find the angle that the second wire makes with the ground, given that it is 20 m long.

16. A surveyor draws a line due west from A to B but cannot continue the line in the westerly direction because of an obstacle. So he draws a line 200 m long from B to C, in a direction 25° west of south and then draws another line in the direction 46° west of north. How long should CD be, if D is to be due west of B ?

17. From an observation tower a metres above the level of a river, the angle of depression of a point on the near shore is α and that of a point directly beyond on the far shore is β. Show that the width of the river is a $(\cot \beta - \cot \alpha)$.

18. From the top of a building a metres in height, on level ground, the angle of depression of the bottom of a vertical tower is α. From the bottom of the

building the angle of elevation of the top of the tower is β. Show that the height of the tower is a $\tan \beta \cot \alpha$.

19. A vertical tower stands on a horizontal plane and is surmounted by a vertical flagstaff of height h. At a point on the plane, the angles of elevation of the bottom and the top of the flagstaff are respectively α and β. Prove that the height of the tower is

$$\frac{h \tan \alpha}{\tan \beta - \tan \alpha}.$$

20. The angles of elevation of the top of a vertical tower from two points distant a and b from the base and in the same straight line with it, are complementary. Prove that the height of the tower is $\sqrt{(ab)}$, and if θ be the angle subtended at the top of the tower by the line joining these points, then $\sin \theta = \dfrac{a - b}{a + b}$.

6.12. Problems Involving Oblique Triangles

Example 19. A tree stands vertically on a hillside which makes an angle of 20° with the horizontal. From a point 35 m directly down the hill from the base of the tree, the angle of elevation of the top of the tree is 43°. Find the height of the tree.

Solution: In the adjoining figure, BC represents the tree, A the point 35 m directly down the hill from the base of the tree, and *a* the height of the tree.

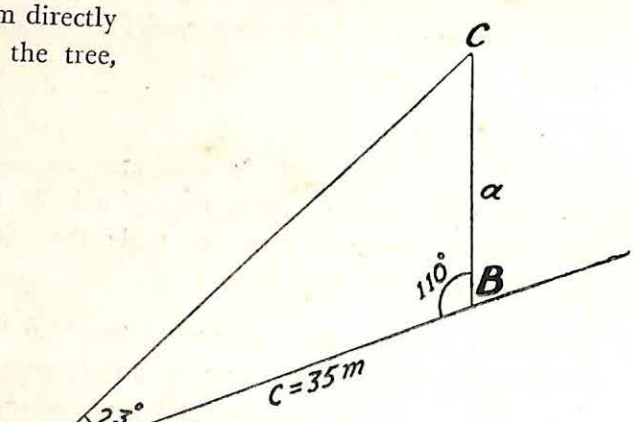


Fig. 6.24

In the triangle ABC, we have

$$\angle \text{BAC} = 43° - 20° = 23°,$$
$$\angle \text{ABC} = 180° - (90 - 20)° = 110°,$$
$$\text{CB} = 180° - (23° + 110°) = 47°,$$
$$c = 35 \text{ m}.$$

By the law of sines,

$$\frac{a}{\sin 23^\circ} = \frac{c}{\sin 47^\circ},$$

$$\text{so that } a = \frac{35 \sin 23^\circ}{\sin 47^\circ} = \frac{35 \times \cdot 391}{\cdot 731} = 19 \text{ m.}$$

Example 20. A surveyor laying a road due west from A encounters a swamp at B. He changes his direction to N 29° W for 2300 metres to C and then turns S 41° W. How far must he continue in this direction to reach a point D on the east-west line through A ?

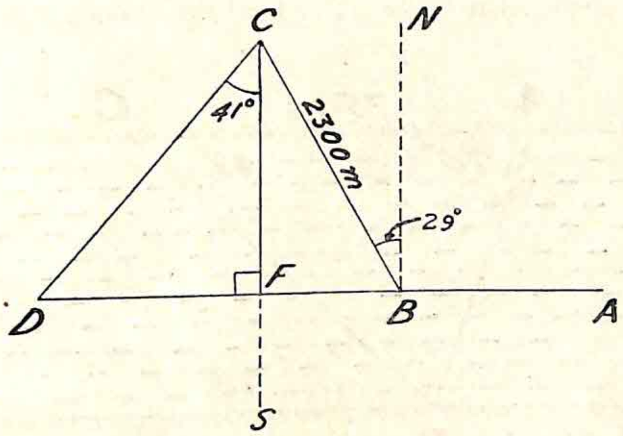


Fig. 6.25

Solution. In the figure above, BC = 2300 m,

$$\angle\, CBD = 90^\circ - 29^\circ = 61^\circ,\ \angle\, CDB = 90^\circ - 41^\circ = 49^\circ,$$

$$\angle\, DCB = 180^\circ - (\,61^\circ + 49^\circ\,) = 70^\circ.$$

By the sine law,

$$\frac{BD}{\sin \angle\, DCB} = \frac{BC}{\sin \angle\, CDB},$$

$$\text{or} \quad BD = \frac{2300 \times \sin 70^\circ}{\sin 49^\circ}$$

$$= \frac{2300 \times \cdot 940}{\cdot 755}$$

$$= 3100 \text{ metres.}$$

EXERCISES

1. Two boats leave a place at the same time. One travels 56 km in the direction N 50° E while the other travels 48 km in the direction S 80° E. What is the distance between the new positions of the boats ?

2. From a defence battery P an enemy ship is sighted in the direction N 57° 20′ E. From a second defence battery Q, 11.4 km due north of P, the ship is seen in the direction S 40° 50′ E. How far is the ship from P ? How far is it from Q ?

3. Two ships leave a port at noon. One ship travels at 16 km per hour in the direction N 13° W. The other ship travels at 21 km per hour in the direction N 57° E. How far apart are they at 2 p.m.?

4. To find the distance between two points A and B on the opposite sides of a river, we measure the distance from A to C to be 75 m, the angle CAB to be 70° and the angle ACB to be 42°. Find the distance AB.

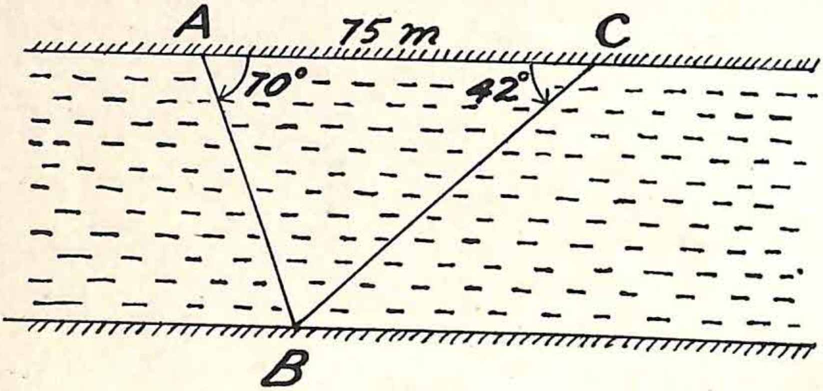


Fig- 6.26

5. Two men, 750 m apart, sight a balloon which is between them and in their vertical plane. The angle of elevation of the balloon as measured by one of the men is 36° and by the other is 56°. Find the distance of the balloon from each observer and the height of the balloon.

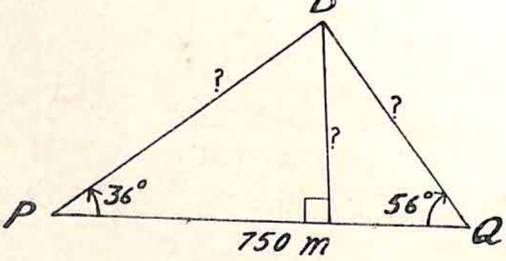


Fig. 6.27

6. A tree stands vertically on a hillside which makes an angle 22° with the horizontal. From a point 20 m directly down the hill from the base of the tree, the angle of elevation of top of the tree is 40°. Find the height of the tree.

7. A tree stands on a hillside which makes an angle of 20° with the horizontal. From a point 50 m up the hill from the tree, the angle of depression of the top of the tree is 10°. Find the height of the tree.

8. A hill slopes at an angle of 18° to the horizontal. A vertical tower stands on the hill. A man walks 75 metres directly down the hill from the foot of the tower and then observes that the angle of elevation of the top of the tower is 50°. Find the height of the tower.

9. Two points A and B are on opposite sides of a pond. The distance from A to a third point C is measured to be 10·5 metres and the distance from C to B is measured to be 13·8 metres. Find the distance from A to B if the angle ACB is 51° 40′.

10. An aeroplane, diving at an angle of 5°, passes directly over an observer on the ground, who finds that in 15 seconds the angle of elevation of the aeroplane alters from 71° to 20°. If the speed of the aeroplane be 660 km p.h., find the height of aeroplane when it is directly overhead.

11. 5-metre pole placed vertically on a hillside casts a shadow of 7 m straight down the slope. At the tip of the shadow, the angle subtended by the pole is 35°. Find the elevation of the sun and the angle made by the hillside with the horizontal.

(Hint: The elevation of the sun is $90° - \alpha$; the angle made by the hillside with the horizontal is $\beta - 90°$).

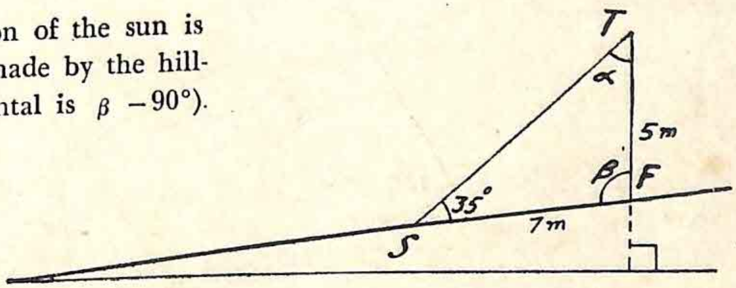


Fig. 6.28

12. On a hillside that makes an angle of 16° with the horizontal, a 10 metre tree leans downhill. When the elevation of the sun is 50°, the shadow of the tree is 5 m long and falls straight up the slope. Find the angle made by the tree with the vertical.

(Hint : The desired angle is $\alpha - 74°$.)

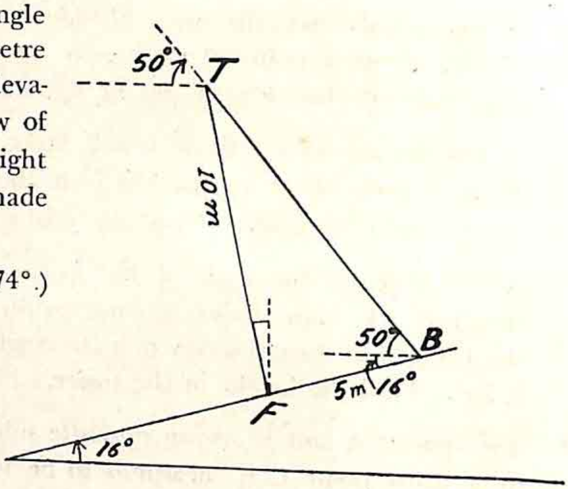


Fig. 6.29

13. A vertical tree stands on a hillside that makes an angle α with the horizontal. From a point directly up the hill, the angle of elevation of the top of the tree is β. From a ponit 1 metre further up the hill, the angle of depression of the top of the tree is γ. If the tree be h metres high, express h in terms of 1, α, β, γ.

14. P, Q, R are three towns. P is 270 km from Q and 180 km from R. The bearing of P from Q is N 18° W. The bearing of R from Q is N 55° W. What can we say about the distance between Q and R ?

15. Find the length x in the following figure :

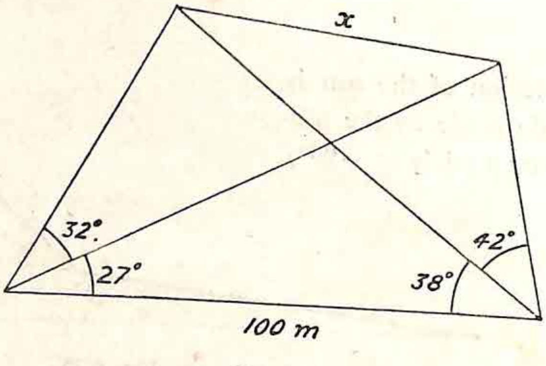


Fig. 6.30

6.13 Three-Dimensional Problems

Example 21. P and Q are two points exactly opposite each other on the two sides of a level road. There is a vertical pillar at P. The angle of elevation of the top of the pillar as measured from Q is 38°. On moving 110 metres along the road to a point R, $\angle$ QRP is found to be 20°. Find (i) the width of the road, (ii) the height of the pillar, (iii) the angle of elevation of the top of the pillar as observed from R.

Remark. The present problem is a three-dimensional problem. In figure 6·31 we have tried to represent this problem on paper (which is two-dimensional). In the figure, three right angles have been represented even though none of them actually measures a right angle. We must remember that when we draw three-dimensional diagrams, the angles and distances may be distorted but vertical lines should always appear as vertical lines. We shall find it helpful to draw right-angled triangles on separate diagrams.

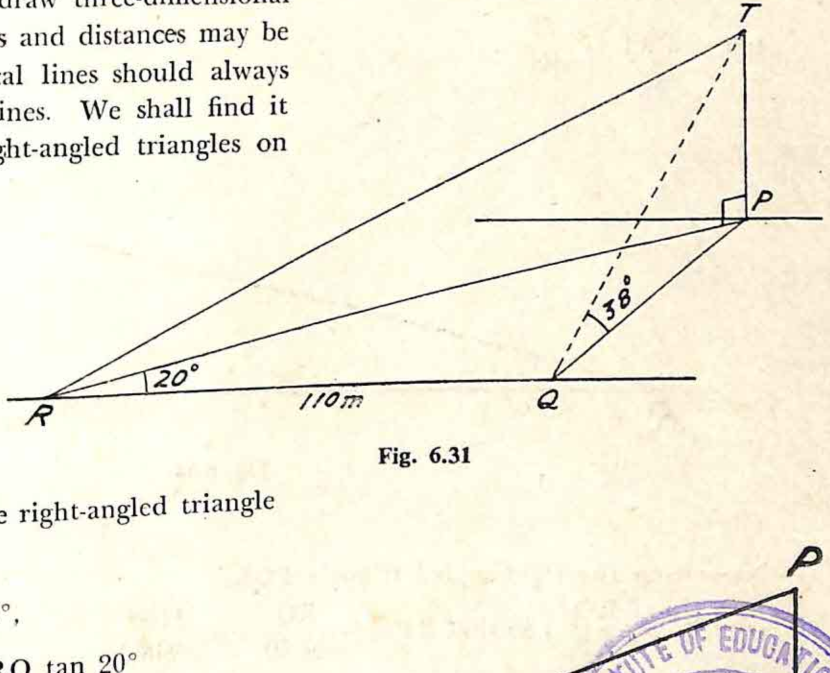


Fig. 6.31

Solution (i) In the right-angled triangle PQR, we have

$$\frac{PQ}{RQ} = \tan 20^\circ,$$

so that $PQ = RQ \tan 20^\circ$

$= 110 \times \cdot 364$

$= 40$ m.

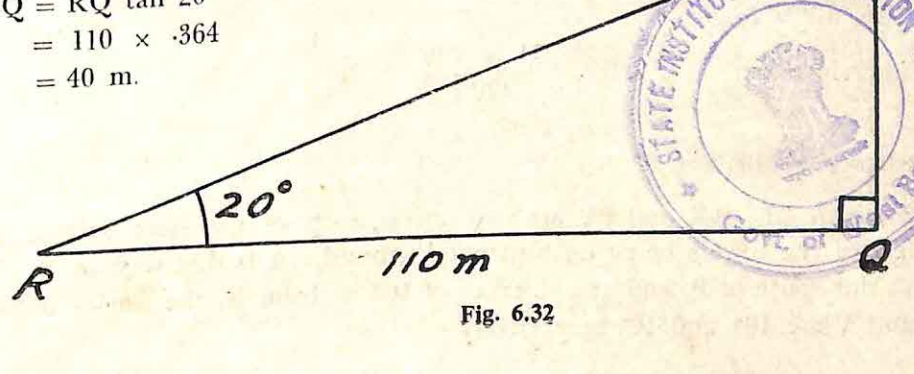


Fig. 6.32

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(ii) In the right-angled triangle TPQ, we have

$$\frac{TR}{PQ} = \tan 38°,$$

so that $TP = PQ \tan 38°$

$= 40 \times 1781$

$= 31$ m.

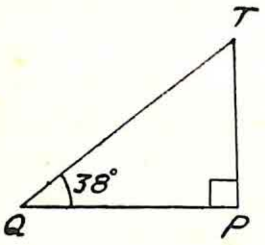

Fig. 6.33

(iii) In the right-angled triangle TPR, we have

$$\tan \angle PRT = \frac{TP}{RP}.$$

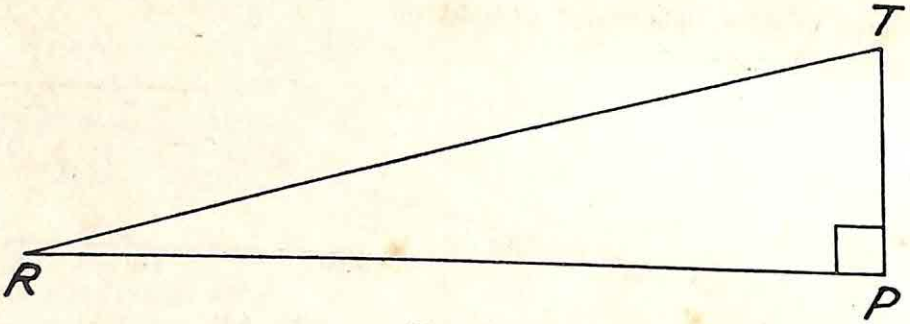

Fig. 6.34

Now from the right-angled triangle PQR,

$$\cos 20° = \frac{RQ}{RP}\text{ ; so that } RP = \frac{RQ}{\cos 20°} = \frac{110}{\cdot 946}.$$

Also, TP = 31 m.

$$\text{Therefore } \tan \angle PRT = \frac{31 \times \cdot 946}{110}$$

$$= \cdot 2666,$$

$$\text{whence } \angle PRT = 15°.$$

Example 22. AX and BY are two towers, each of the same height, A and B, the feet of the towers being on horizontal ground. A is due west of B. From a point C, due south of B, and at a distance of 100 m from it, the angles of elevation of X and Y are 10° and 12° respectively.

Find (i) the height of each pillar ;
(ii) the distance CA ;
(iii) the distance AB between the feet of the Towers.

Solution. The diagram is shown in figure 6.35. Let us note that ∠ XAC, ∠ YBC, ∠ ABC are all right angles even though they do not appear to be so.

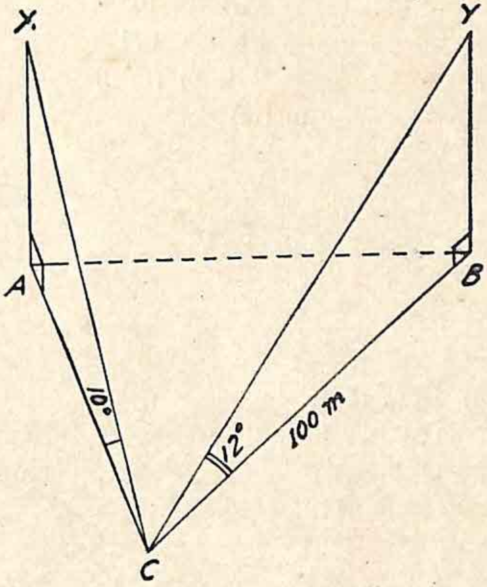

Fig. 6.35

(i) From the right-angled triangle YBC,
we have $\tan 12° = \frac{YB}{CB}$,
so that $YB = CB \tan 12°$
$= 100 \times \cdot 2126$
$= 21$ m. (to the nearest metre)

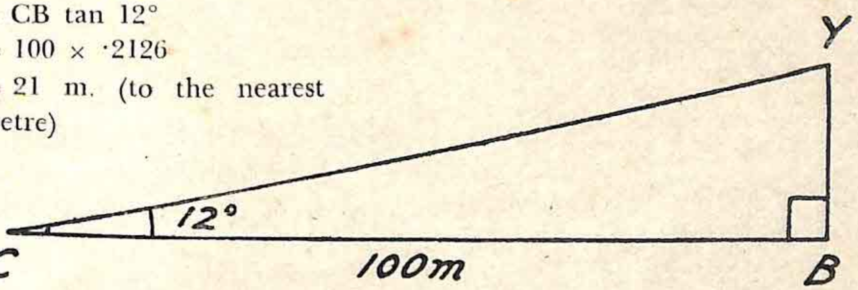

Fig. 6.36

Hence the height of each tower is 21 metres.

(ii) From the right-angled triangle CAX,

we have $\tan = \frac{XA}{CA}$,

so that $CA = XA \cot 10°$

$= 21 \times 5{\cdot}671$

$= 119$ m (to the nearest metre)

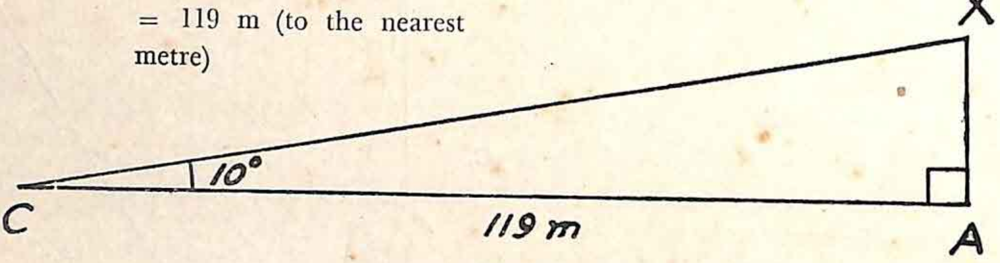

Fig. 6.37

(iii) From the right-angled triangle ABC, we have

$AB^2 = AC^2 - BC^2$

$= (119)^2 - (100)^2$

$= 4161,$

so that $AB = \sqrt{4161} = 65$m.

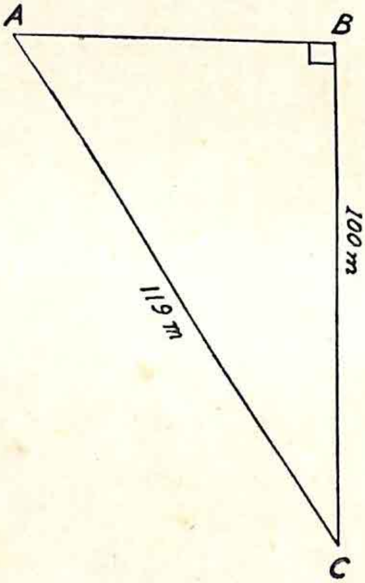

Fig. 6.38

EXERCISES

1. The angle of elevation of a tower from the point A due south of it is 42° and from a point B due east of A is 27°. If AB = 45 metres, find the height of the tower.
2. From A, a point P bears due west and its elevation is 27°. From B, 500 metres south of A, and at the same level as A, the elevation of P is 14°. Find the height of P above the level of A.

3. P and Q are two points on the same level as the base F of a vertical pillar 29 metres high, and the angle of elevation of the top of the pillar when viewed from either point is 26°. If $\angle$ XFY is 90°, find the angle of elevation when the pillar is viewed from the point on PQ which is nearest to F.

4. Three equal poles, 30 metres high, are placed at equal intervals on the same side of a road 10 metres wide. An observer stands at O, on the other side of the road, directly opposite the first pole. He finds that the angle of elevation of the top of the second pole is 27°. Find (i) the distance between each pair of poles; (ii) the angle of elevation of the top of the third pole from O.

5. From the top of a cliff 60 metres high, two ships are seen, one due south at an angle of depression 12°, the other in a direction S 64° W at an angle of depression of 10°. Find the distance between the ships.

6. An observer at A looks due north and sees a meteor with an angle of elevation 68°. At the same time another observer 48 km east of A, approximates the position of the meteor as N 50° W but does not note its angle of elevation. Find the height of the meteor and its distance from A.

7. The angle of elevation of the top of a vertical tower from a point A due south of it is known to be θ and from a point due west of A to be ϕ. If AB = a, show that the height of the tower is

$$\frac{a \sin \theta \sin \phi}{\sqrt{\{\sin(\theta + \phi)\sin(\theta - \phi)\}}}.$$

8. At each end of a base of length 2a, the elevation of the top of a mountain is α, and at the middle point of the base, the elevation is β. Prove that the height of the mountain is

$$\frac{a \sin \alpha \sin \beta}{\sqrt{\{\sin(\alpha + \beta)\sin(\alpha - \beta)\}}}.$$

9. The altitude of the top of a mountain observed from each of three points A, B, C, forming a triangle on a plane, is α. Show that its height is $(1/2)$ a $\tan \alpha$ cosec A.

10. A person, wishing to find the height of an inaccessible vertical pillar OP standing on a horizontal plane, observes that at a point A in the plane the angle of elevation of the top P of the pillar is α. After walking a distance a feet to a point B, $\angle$ PAB being β, he finds that $\angle$ PBA is γ. Find the height of the pillar.

ASSORTED EXERCISES ON CHAPTER 6

1. Solve, without using logarithms, the triangles in the following problems :

(i) $a = 16$, $b = 18$, $c = 25$.
(ii) $a = 26$, $b = 43$, $c = 31$.
(iii) $a = 70$, $b = 50$, $C = 24°$.
(iv) $a = 7{\cdot}4$, $B = 47°$, $C = 54°$.
(v) $a = 25$, $b = 12$, $A = 51°$.
(vi) $a = 46$, $b = 51$, $A = 62°$.
(vii) $b = 40$, $c = 24$, $B = 100°$.
(viii) $c = 2{\cdot}5$, $a = 3{\cdot}7$, $C = 29°$.

2. Solve the triangles in the following problems by using logarithms and check the solution in each case :

(i) $a = 99{\cdot}93$, $b = 92{\cdot}04$, $c = 81{\cdot}05$.
(ii) $a = 563$, $b = 462$, $C = 40° \ 20'$.
(iii) $a = 83{\cdot}21$, $B = 47° \ 41'$, $C = 61° \ 29'$.
(iv) $b = 36{\cdot}91$, $c = 24{\cdot}68$, $B = 137° \ 41'$.
(v) $a = 57{\cdot}14$, $c = 38{\cdot}48$, $C = 35° \ 30'$.
(vi) $a = 326{\cdot}5$, $c = 529{\cdot}3$, $A = 32° \ 47'$.
(vii) $b = 42{\cdot}38$, $c = 48{\cdot}86$, $B = 21° \ 9'$.

3. The sides of triangle are $4\sqrt{3}$, 7, $\sqrt{13}$ centimetres respectively. Find its smallest angle.

4. Show that a, b, $\sqrt{\{a^2 + b^2 + ab\}}$ determine a triangle for each pair of positive values of a and b. Find the greatest angle of any such triangle.

5. The three sides of a triangle are in A.P. and the greatest angle exceeds the least by a right angle. Prove that the sides are in the ratios

$$\sqrt{7} + 1 : \sqrt{7} : \sqrt{7} - 1.$$

6. Show that if the cosines of two angles of a triangle be directly proportional to the opposite sides, the triangle is isosceles ; but if they are inversely proportional to the opposite sides, the triangle is either isosceles or right-angled.

7. Two points, A and B are 186 metres apart on a level stretch of road leading to a hill. The angle of elevation of the hill-top from A is 25° 40′, and the angle of elevation from B is 37° 30′. Find how high the hill extends above AB produced.

8. A ship is being tracked from two stations A and B 4500 metres apart on a west-east line. At A the ship is observed in the direction S 53° E while at B it is in the direction S 41° E. How far is the ship from B ?

9. A vertical tree stands on a hillside that is inclined at an angle of 12° with the horizontal. When the angle of elevation of the sun measures 27°, the shadow of the tree down the slope is 13 m long. How tall is the tree ?

10. If the angle of elevation of a cloud from a point h metres above a lake be β, and the angle of depression of its reflection in the lake α, prove that the height of the cloud is

$$h\,\frac{\sin(\alpha-\beta)}{\sin(\alpha+\beta)}.$$

11. At two points A and B 250 metres apart, on a straight horizontal road, the summit of a hill is observed. At A it is due north, with an elevation of 40°. At B it is due west, with an elevation of 28°. Find the height of the hill.

12. Two stations due south of a tower, which leans towards the north, are at distances a and b metres from its foot. If α and β are the elevations of the top of the tower from these stations, prove that its inclination to the horizontal is $\cot^{-1}\dfrac{b\cot\alpha - a\cot\beta}{b-a}$.

13. From a mountain village the angle of elevation of a mountain peak is α. After leaving the village and walking a distance d down a slope that makes an angle β with the horizontal, an observer finds the angle of elevation of the peak to be γ. Express the difference in the altitudes of the peak and the village in terms of d, α, β and γ.

14. A vertical television mast is mounted on the roof of a building. From a point 260 m from and on a level with the base of the building, the angles of elevation of the base and top of the mast are 36° and 52° respectively. How tall is the mast ?

15. An aeroplane flying horizontally at 3000 m is observed due west at an elevation of 21°. One minute later it is seen due south at an elevation of 51°. Calculate (a) the speed of the aeroplane in km. p. h. and (b) the direction of aeroplane's flight.

16. The angle of elevation of a tower from a point A due south of it is x and from a point due east of A is y. If AB = 1, show that the height h of the tower is given by $h^2(\cot^2 y - \cot^2 x) = 1^2$.

Chapter 7

Trigonometric Functions of A Real Variable

7.1. Introduction

We have defined a trigonometric function as a mapping of the set of all (or some) angles into the set of real numbers, i.e., we have defined trigonometric functions of angles. We now propose to define trigonometric functions of real numbers, the domain in each case being a suitable subset of **R**. To this end, we shall first introduce the idea of the composite of two functions.

7.2. Composite of Two Functions

Let f be a function with domain X and range Y, and let g be a function with domain Y and range Z.

Let us define a function h with domain X and range Z by setting

$$h(x) = g\ (f(x)) \text{ for all } x \in X.$$

The function h thus defined is called the composite of the functions f and g, and is denoted by $g^\circ f$.

To find the image of any element x of X under the function $g^\circ f$, we have to take first its image under f, and then take the image of f(x) under g. Since f(x) is an element of the domain of g. it is meaningful to talk of g (f(x)). Also, g (f(x)) is obviously an element of Z, the range of g.

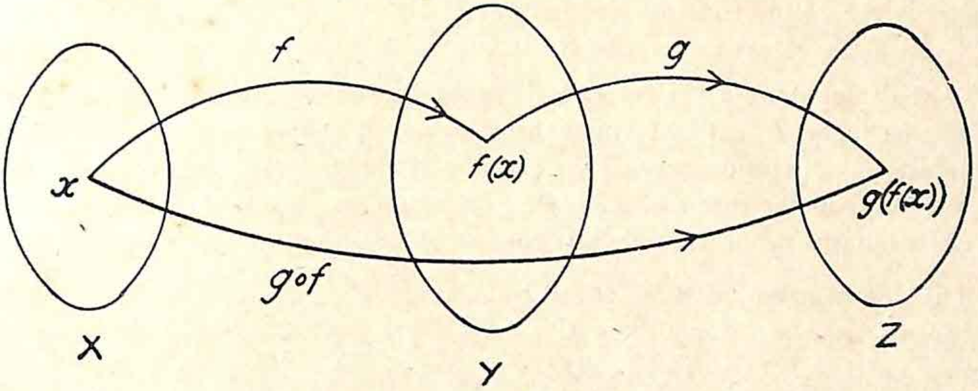

Fig. 7.1

Let us note that we have defined g° f for the case when the domain of g is the same as the range of f. (More generally, one can define g°f when the range of f is contained in the domain of f.)

Given two functions f and g, it may happen that :

(i) None of g° f and f° g exists;

(ii) g° f exists but f° g does not exists;

(iii) f° g exists but g° f does not exist;

(iv) f° g and g° f both exist but are not equal;

(v) f° g and g° f both exist and are equal.

7.3. Trigonometric Functions of Real Numbers

Let f be the function with domain **R**, the set of all real numbers and range **a**, the set of all angles, which maps each real number x into an angle whose measure in

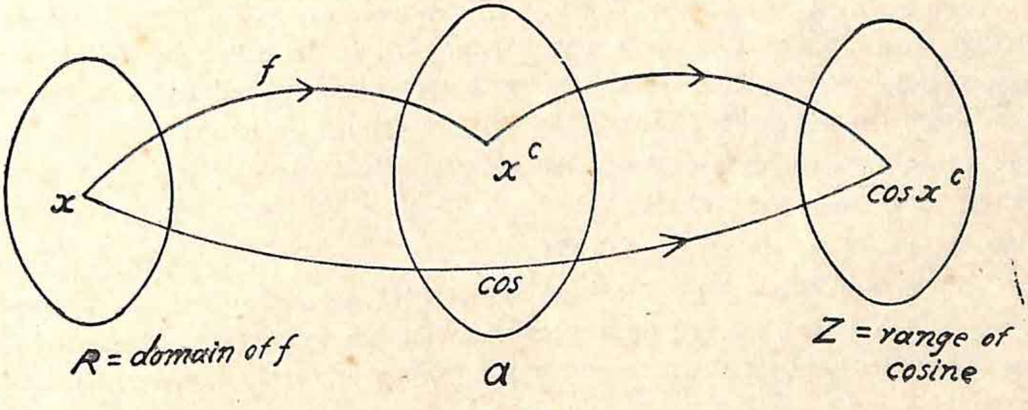

Fig. 7.2

radians is x^c. Also let g be the cosine function which maps each angle into its cosine. We can then talk of the composite function $g^\circ f$. The function $g^\circ f$ maps each real number x into the cosine of the angle whose measure in radians is x^c. This function is the cosine function defined on **R**. We write

$$\cos x = \cos x^c, \text{ for all } x \in \mathbf{R}.$$

Use of the same symbol cos for two different functions—the cosine function on **Q** defined in chapter 2, and the cosine function on **R** defined just now—is not likely to produce any confusion because we can always judge from the context whether we are talking of the cosine of an angle or that of the cosine of a number.
We can similarly define the other trigonometric functions of real numbers.

The sine function on **R** is defined by setting

$$\sin x = \sin x^c \text{ for all } x \in \mathbf{R}.$$

Thus $\cos = \{(x, \cos x^c) : x \in \mathbf{R}\}$,

$\sin = \{(x, \sin x^c) : x \in \mathbf{R}\}$.

The domain of each of the functions cos and sin is the set **R** of all real numbers.

Denoting by $\mathbf{R}^*$ the set $\mathbf{R} \sim \{2k + 1)\pi/2 : k \text{ is an integar}\}$, (so that $\mathbf{R}^*$ is the set of all those real numbers that are not odd multiples of $\pi/2$), we can define the tangent and the secant functions on $\mathbf{R}^*$ corresponding to the tangent and secant functions defined on **Q** in chapter 2.

The tangent function on $\mathbf{R}^*$ is defined by setting

$$\tan x = \tan x^c, \text{ for all } x \in \mathbf{R}^*.$$

The secant function on $\mathbf{R}^*$ is defined by setting

$$\sec x = \sec x^c, \text{ for all } x \in \mathbf{R}^*.$$

Thus $\tan = \{(x, \tan x^c) : x \in \mathbf{R}^*\}$

$\sec = \{(x, \sec x^c) : x \in \mathbf{R}^*\}$.

Denoting by $\mathbf{R}^{**}$ the set $\mathbf{R} \sim \{k\pi : k \text{ is an integar}\}$, (so that $\mathbf{R}^{**}$ is the set of all those real numbers that are not multiples of π), we can define the cotangent and the cosecant functions on **R** corresponding to the cotangent and the cosecant functions defined on **Q** earlier. These functions are defined by setting

$$\cot x = \cot x^c, \text{ for all } x \in \mathbf{R}^{**}.$$

$$\csc x = \csc x^c, \text{ for all } x \in \mathbf{R}^{**}.$$

Thus $\cot = \{(x, \cot x^c) : x \in \mathbf{R}^{**}\}$,

$\csc = \{(x, \csc x^c) : x \in \mathbf{R}^{**}\}$.

It can be easily seen that all the properties established so far for trigonometric functions of angles hold for the corresponding functions of real numbers. Thus, for example

$$\left.\begin{array}{l}\cos(\pi/2 - x) = \sin x,\\ \cos(2\pi + x) = \cos x,\end{array}\right\}\text{ for all } x \in \mathbf{R}$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y,$$

for all $x, y \in \mathbf{R}$.

7.4. Bounded and Unbounded Trigonometric Functions

Let us recall that a real valued function f is said to be bounded if we can find a number M such that

$$f(x) \mid \leqslant M, \text{ for all } x \text{ in the domain of f.}$$

for all $x, y \in \mathbf{R}$.

$$\cos^2 x + \sin^2 x = 1,$$

so that $0 \leqslant \cos^2 x \leqslant 1$ and $0 \leqslant \sin^2 x \leqslant 1$,

or $|\cos x| \leqslant 1$ and $|\sin x| \leqslant 1$, for all $x \in \mathbf{R}$

therefore cos and sin are bounded functions.

We shall see later that the functions tan, cot, sec and csc are not bounded.

7.5. Intervals on which a Trigonometric Function has a Fixed Sign

From Art. 2.3, we have the following :

1. If the terminal side of an angle lies in the right-half plane, its cosine and secant are positive ; in terms of numbers this means that *for each integer k, cos and sec are positive in the open interval* $]-\pi/2+2k\pi, \pi/2+2k\pi[$. Similarly for each integer k, cos and sec are negative in the open interval $]\pi/2+2k\pi, 3\pi/2+2k\pi[$.
2. If the terminal side of an angle lies in the upper half-plane then the sine and the cosecant of the angle are positive and if it lies in the lower half-plane, the sine and the cosecant of the angle are negative. This means that *for each integer k, sin and csc are positive in the open interval* $]\, 2k\pi, (2k + 1)\pi\, [$ *and negative in the open interval* $]\,(2k + 1)\pi, (2k + 2)\pi\, [$.
3. If the terminal side of an angle lies in the first or the third quadrant, the tangent and the cotangent of the angle are positive, and if the terminal side lies in the second or the fourth quadrant, the tangent and the cotangent of the angle are negative. This means that *for each integer k, tan and cot are positive in the interval* $]\, k\pi, k\pi+\pi/2\, [$, *and negative in the interval* $]\, k\pi - \pi/2, k\pi\, [$.

7.6. Periodicity of Trigonometric Functions

Let f be a function whose domain D is a sub-set of **R**.

A real number p ($\neq 0$) is said to be a *period* of f if

(i) $x \in D \Longleftrightarrow x + P \in D$;

(ii) $f(x + p) = f(x), \ \forall\, x \in D$.

For a constant function, every non-zero number is a period.

A function f is said to be *periodic* if it has at least one period.

It is obvious that if p is a period, then every positive integral multiple of it is also a period :

i.e., $f(x + kp) = f(x), \quad \forall\, k \in \mathbf{Z}^{+}$.

From condition (i) in the definition of a period, it follows that if x is in the domain of a periodic function with p as a period, then so also is $x - p$, for

$$x = (x - p) + p, \text{ so that}$$
$$x \in D \Longleftrightarrow (x - p) \in D.$$

Also, by condition (ii), we then have

$$f(x) = f(x - p + p) = f(x - p).$$

Therefore $-p$ is also a period. Since every positive integral multiple of a period is also a period, therefore $-kp$ is also a period for each positive integer k.

Thus, if p is a period, then kp (where k is any integer) is also a period. The least positive period of a periodic function is said to be the *period* of the function.

Periodic functions are of great importance in physics, mechanics and engineering for the study of various periodic phenomena such as vibration, motion of machines, alternating electric currents, etc. As we shall presently see, the trigonometric functions are periodic and one of the most important reasons for the study of trigonometric functions is their periodicity.

Theorem. All trigonometric functions are periodic. Also,

(a) the period of the functions cos, sin, sec, csc is 2π ?

(b) the period of the functions tan and cot is π.

Proof. (i) Since $\cos x = \cos(x + 2\pi), \ \forall\, x \in R$, 2π is a period of the cosine function. We shall next show that it is the *period*, i.e., it is the least positive period. If not, let a positive number $p < 2\pi$ be a period. Then $\cos(x + p) = \cos x$, for all $x \in R$. Putting $x = 0$ in this relation we have $\cos p = 1$. Since $0 < p < 2\pi$, the terminal side of the angle having a measure p radians cannot coincide with the positive x-axis and consequently $\cos p \neq 1$. This gives a contradiction. Hence 2π is the period of the cosine function.

(ii) Since $\sin x = \sin(x + 2\pi), \ \forall\, x \in R$, 2π is a period of the sine function. We shall now show that 2π is the least positive period. If not, let a positive number $p < 2\pi$ be a period. Then $\sin(x + p) = \sin x$ for all x. Putting $x = \pi/2$ in this relation, we have $\sin(\pi/2 + p) = \cos p = 1$; but this is impossible since $0 < p < 2\pi$. Hence 2π is the period of the sine function.

(iii) Since $\tan(x + \pi) = \tan x$ for all $x \in R$, π is a period of the tangent function. We shall now show that π is the period. If not, let a positive number $p < \pi$ be a period. Then $\tan(x + p) = \tan x$ for all $x \in R$. Putting $x = 0$ in this relation, we have $\tan p = 0$ which is impossible since $0 < p < \pi$. Hence π is the period of the tangent function.

The rest of the theorem can be proved in the same manner as above.

The property of periodicity simplifies the study of a trigonometric function because by studying the properties of such a function over an interval of length equal to the period, we can know its properties everywhere in the domain of the function.

7.7. Intervals of Monotony of Trigonometric Functions

A function f is said to be *strictly increasing* in an interval if for any two numbers x_1, x_2 in the interval

$$x_2 > x_1 \Rightarrow f(x_2) > f(x_1).$$

A function f is said to be *strictly decreasing* in an interval if for any two numbers x_1, x_2 in the interval,

$$x_2 > x_1 \Rightarrow f(x_2) < f(x_1).$$

A function f is said to be *strictly monotonic* if it is either strictly increasing or strictly decreasing.

Note. If a function f has the property that for any numbers x_2, x_1 in an interval, $x_2 > x_1 \Rightarrow f(x_2) \geq f(x_1)$, then we say that f is *increasing* in the interval. Similarly we speak of *decreasing* and monotonic functions.

Examples. (i) Let f be the function defined by setting $f(x) = x^2 \ \forall\ x \in R$. Then f is strictly increasing in $[0, 1]$ and strictly decreasing in $]-1, 0[$.

(ii) Let f be the function defined by setting

$$f(x) = \begin{cases} 0 & \text{if } x \leqslant 0 \\ x^2 & \text{if } x > 0, \end{cases}$$

Then f is increasing, but not strictly increasing, in $[-1, 1]$.

However, f is strictly increasing in $[0, 1]$.

(iii) Let f be the function defined by setting

$$f(x) = x^2 - 1, \ \forall\ x \in R.$$

Then f is neither increasing nor decreasing in $[-1, 1]$.

We now propose to determine the intervals in which a given trigonometric function is monotonic.

The function cos. Let x_1, x_2 be two real numbers such that $x_2 > x_1$.

We have $\cos x_2 - \cos x_1 = -2 \sin \frac{1}{2}(x_2 - x_1) \sin \frac{1}{2}(x_2 + x_1)$. ....(1)
If x_1, x_2 be both in the interval $]0, \pi[$, then $\frac{1}{2}(x_2 + x_1)$ is also in the same interval and so also is $\frac{1}{2}(x_2 - x_1)$, and consequently $\sin \frac{1}{2}(x_2 - x_1)$ and $\sin \frac{1}{2}(x_2 + x_1)$ are both positive. Therefore from (1) we find that $\cos x_2 - \cos x_1 < 0$, i.e., $\cos x_2 < \cos x_1$. Since for all x_1, x_2 in $]0, \pi[$, $x_2 < x_1 \Rightarrow \cos x_2 < \cos x_1$, therefore in the interval $]0, \pi[$, cos is a strictly decreasing function. Also as $x = 0$, the cosine function has the value 1 which is the greatest value that cos x can have for any value of x, and at $x = \pi$, the cosine function has the value -1 which is the least value that cos x can have for any value of x. Thus on the closed intervals $[0, \pi]$, cos strictly decreases from 1 to -1.

As $\cos(\pi + x) = -\cos x$, it follows that cos is a strictly increasing function in $[\pi, 2\pi]$.

Since the cosine function is periodic with period 2π, therefore it will have the same behaviour in all intervals of the form $[2k\pi, (2k+1)\pi]$, k being any integer, as in $[0, \pi]$.

Similarly it will have the same behaviour in all intervals of the form $[(2k+1)\pi, (2k+2)\pi]$, k being an integer, as in $[\pi, 2\pi]$.

Thus *for each integer k, cos x strictly decreases from +1 to −1 on the interval* $[2k\pi, (2k+1)\pi]$, *and strictly increases from −1 to +1 on the interval* $[(2k+1)\pi, (2k+2)\pi]$.

Thus *for the cosine function the intervals of strict decrease are* $[2k\pi, (2k+1)\pi]$ *and the intervals of strict increase are* $[(2k+1)\pi, (2k+2)\pi]$.

Since $\sin x = \cos(x - \pi/2)$, therefore from the results obtained above in respect of the function cos, it follows that sin x strictly decreases from 1 to -1 as $x - \pi/2$ increases from 0 to π, and strictly increases from -1 to $+1$ as $x - \pi/2$ increases from π to 2π; i.e., sin x strictly decreases from 1 to -1 as x increases from $\pi/2$ to $3\pi/2$, and strictly increases from -1 to 1 a sx increases from $3\pi/2$ to $5\pi/2$. By periodicity, the behaviour of the sine function in the interval $[\pi/2, 3\pi/2]$ is the same as that in the interval $[2k\pi + \pi/2, 2k\pi + 3\pi/2]$, k being any integer.
Thus *for each integer k, sin x strictly decreases from 1 to −1 on the interval* $[2k\pi + \pi/2, 2k\pi + 3\pi/2]$.

Similarly we can show that *for each integer k, sin x strictly increases from −1 to 1 on the interval* $[2k\pi - \pi/2, 2k\pi + 3\pi/2]$.

Thus *for the sine function the intervals of strict decrease are* $[2k\pi + \pi/2, 2k\pi + 3\pi/2]$, *and the intervals of strict increase are* $[2k\pi - \pi/2, 2k\pi + \pi/2]$, *k being any integer.*

The function tan. Let us consider two numbers x_1 and x_2 in $\mathbf{R}^*$ such that $x_2 > x_1$. We have

$$\tan x_2 - \tan x_1 = \frac{\sin (x_2 - x_1)}{\cos x_2 \cos x_1} \cdot$$

If $-\pi/2 < x_1 < x_2 < \pi/2$, then $0 < x_2 - x_1 < \pi$, so that $\sin (x_2 - x_1) > 0$, $\cos x_1 > 0$, $\cos x_2 > 0$. Therefore $\tan x_2 > \tan x_1$, i.e., tan is strictly increasing in $]-\pi/2, \pi/2[$.

As tan is periodic with period π, therefore *for each integer k, tan is strictly increasing in the interval* $]k\pi - \pi/2, k\pi + \pi/2[$.

The functions cot, sec and csc. It can be easily proved that

(i) *for each integer k, cot is strictly decreasing on the interval* $]k\pi, (k+1)\pi[$;

(ii) *for each integer k, sec is strictly decreasing on each of the intevals* $]2k\pi - \pi/2, 2k\pi[$ *and* $]2k\pi + \pi, 2k\pi + 3\pi/2[$ *and is strictly increasing on each of the intervals* $]2k\pi, 2k\pi + \pi/2[$ *and* $]2k\pi + \pi/2, 2k\pi + \pi[$;

(iii) *for each integer k, csc is strictly decreasing on each of the intervals* $]2k\pi, 2k\pi + \pi/2[$ *and* $]2k\pi + 3\pi/2, (2k+2)\pi[$, *and is strictly increasing on each of the intervals* $]2k\pi + \pi/2, 2k\pi + \pi[$ *and* $]2k\pi + \pi, 2k\pi + 3\pi/2[$.

7.8. Ranges of Trigonometric Functions

cos and sin. We have seen that for all $x \in \mathbf{R}$, $-1 \leqslant \cos x \leqslant 1$, so that the range of cos is contained in the interval $[1-, +1]$. We shall now show that the range is actually this interval. To this end we must show that every number in this interval is a value of the function.

Let p be any real number such that $-1 \leqslant p \leqslant 1$. Let us choose $q \geq 0$ such that $p^2 + q^2 = 1$. The point $P(p, q)$ lies on the unit circle.

Let θ be the measure of angle XOP (having OX as the initial side and OP as the terminal side) in radians. Then $\cos \theta = a$.

Thus range of cos is $[-1, 1]$.

Since $\sin x = \cos (\pi/2 - x)$, $\cos x = \sin (\pi/2 - x)$, therefore it follows that the range of sin is also $[-1, 1]$.

tan and cot. Let a number $k \geq 0$ be given. Then $0 \leqslant \frac{1}{\sqrt{(k^2+1)}} \leqslant 1$.

Since the range of cos is the interval $[-1, 1]$, therefore we choose a real number x such that $0 \leqslant x < \pi/2$ and $\cos x = \frac{1}{\sqrt{(1+k^2)}}$ ·

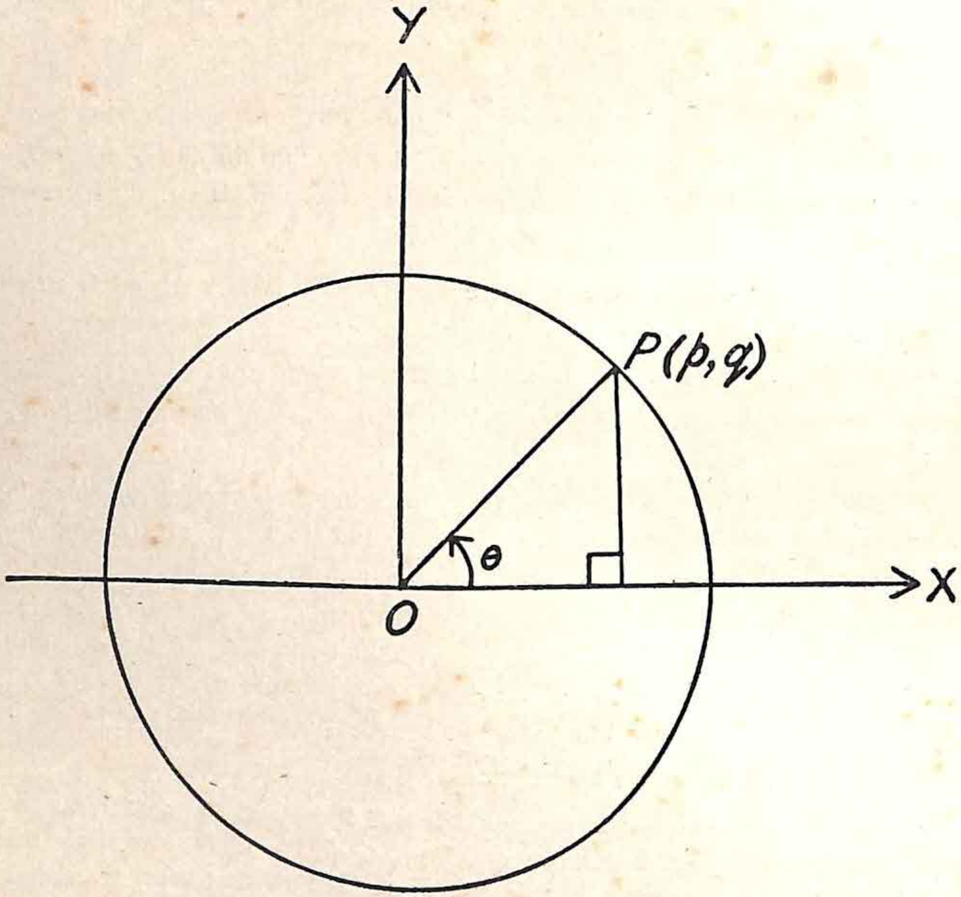

Fig. 7.3 (a)

Since $\sin^2 x + \cos^2 x = 1$, and for all x in $[0, \pi/2[$, sin x is non-negative, therefore it follows that $\sin x = \dfrac{k}{\sqrt{(1+k^2)}}$.

Then $\tan x = k$.

Thus for every $k \geq 0$ we can find an $x \in [0, \pi/2[$ such that $\tan x = k$. Thus every non-negative real number is in the range of tan. Again, since $\tan(-x) = -\tan x$, therefore it follows that every negative number is also in the range of tan.

Hence *the range of tan is the set* **R** *of all real numbers.*

$$\text{Since } \tan x = \cot(\pi/2 - x),$$
$$\text{and } \cot x = \tan(\pi/2 - x),$$

therefore cot has also the same range as tan, i.e., *the range of cot is the set* **R** *of all real numbers.*

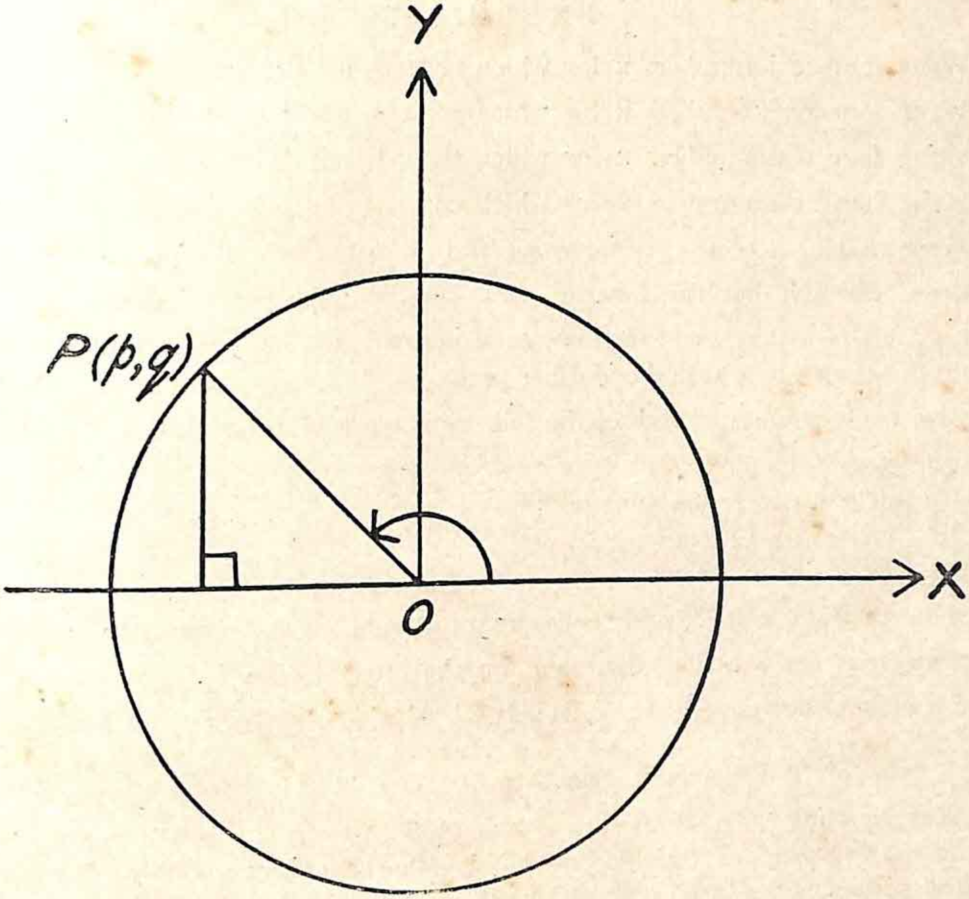

Fig. 7.3 (b)

Corollary. The functions tan and cot are not bounded.
Proof. · Let M be any real number whatever,. Since the range of tan is **R**, therefore we can find a real number x such that $\tan x = |M| + 1$.

Then
$$\begin{aligned} |\tan x| &= \big|\, |M| + 1 \,\big|, \\ &= |M| + 1, \\ &> |M|, \\ &\geq M. \end{aligned}$$

Thus there does not exist any real number M with the property that $|\tan x| \leqslant M$ for all x in the domain of tan. Hence tan is not bounded.

Similarly we can show that cot is not bounded.
sec and csc. We can show in the same manner as above that the range of sec and csc is the set $\{x : x \leqslant -1\} \cup \{x : x \geq 1\}$.

EXERCISES

1. Write four real numbers x for which tan x is not defined.
2. Write four real numbers x for which cot x is not defined.
3. Write four real numbers x for which sec x is not defined.
4. Write four real numbers x for which csc x is not defined.
5. Prove that each of the functions sec and csc is periodic, the period being 2π.
6. Prove directly that the function cot is periodic, the period being π.
7. Use the fact that the function tan is periodic and has period π to prove that the function cot is periodic and has period π.
8. Give two examples of each of the following types of functions:
   (i) strictly increasing function ;
   (ii) strictly decreasing function ;
   (iii) increasing function ;
   (iv) decreasing function.
9. Show that sin x strictly increases from -1 to 1 on the interval $[-\pi/2, \pi/2]$.
10. Show that cot x strictly decreases on the interval $]0, \pi[$.
11. Show that the range of sec is $\mathbf{R} \backsim \,]-1, 1[$.

7.9. Graphs of Trigonometric Functions

The sine function. To draw the graph of sin we use the equation $y = \sin x$, x being the independent variable and y being the dependent variable. We give to x several values in the interval $[0, 2\pi]$ and find the corresponding values of y. The following table shows, correct to two places of decimals, the values of x from 0 to 2π by steps of $\pi/6$.

x	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π	$7\pi/6$	$4\pi/3$	$3\pi/2$	$5\pi/3$	$11\pi/6$	2π
sin x	0	·5	·87	1	·87	·5	0	−·5	−·87	−1	−·87	−·5	0

We choose a suitable unit of length and then plot the points (0, 0), $(\pi/6, \cdot 5)$......, $(2\pi, 0)$. We then join these points by a smooth curve to obtain a part of the graph (figure 7.4). Since the sine function is periodic with period 2π, therefore this part can be reproduced in the intervals $[-2\pi, 0]$, $[2\pi, 4\pi]$, etc. The complete graph is a wave from $x = 0$ to $x = 2\pi$ repeated infinitely many times to the left and to the right.

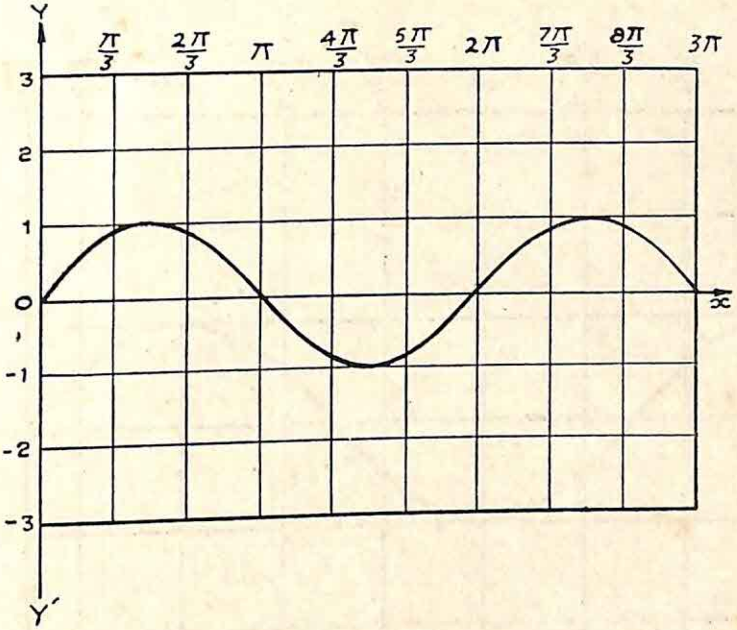

Fig. 7.4 Graph of sin (incomplete graph)

Remark. The graph of the sine function of angles can be drawn in a similar manner.

The cosine function. To draw the graph of cos we begin, as in the case of sin, by preparing a table of values giving values of cos x as x varies from 0 to 2π in steps of $\pi/6$.

x	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π	$7\pi/6$	$4\pi/3$	$3\pi/2$	$5\pi/3$	$11\pi/6$	2π
cos x	1	·87	·5	0	−·5	−·87	−1	−·87	−·5	0	·5	·87	1

By plotting the points (0, 1)……, (2π, 1) and joining them by a smooth curve, we obtain a part of the graph (figure 7.5). Since the cosine function is periodic with period 2π, therefore this part can be reproduced in the interval $[-2\pi, 0]$, $[2\pi, 4\pi]$, etc. The complete graph is a wave from $x = 0$ to $x = 2\pi$ repeated infinitely many times to the left and to the right.

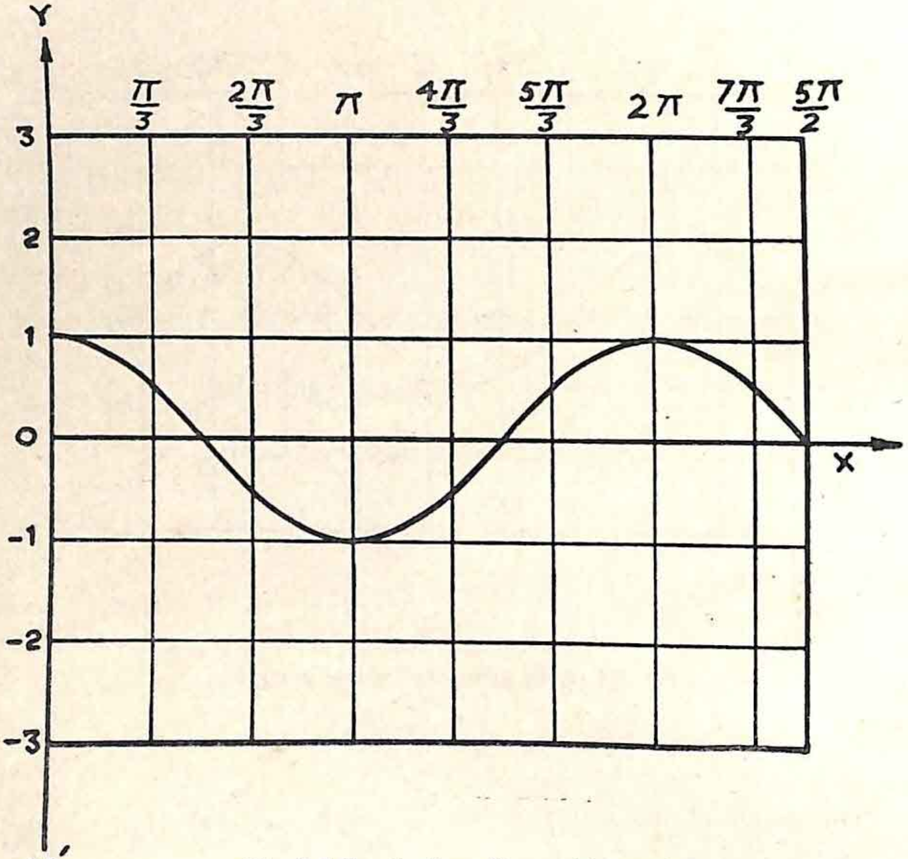

Fig. 7.5 Graph of cos (incomplete graph)

The following figure shows the graphs of sin and cos drawn together for comparison.

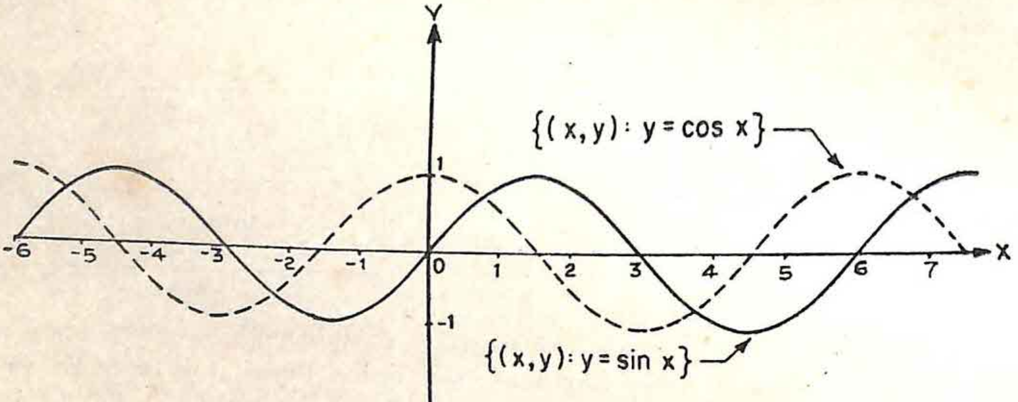

Fig. 7.6 Graphs of sin and cos (incomplete graphs)

From the graphs we can see that a translation (sliding) of the graph of cos through a distance $\pi/2$ units along the positive direction of the x-axis will make it coincide with the graph of sin. This is only to be expected in view of the identity $\sin(\pi/2 + x) = \cos x$.

The tangent and the cotangent functions. To draw the graph of tan, let us first observe that since the tangent function is periodic with period π, therefore it is enough to draw the graph over an interval of length π. The complete graph will then consist of infinitely many repetitions of the same to the left as well as to the right.

Also, since $\tan(-x) = -\tan x$, therefore if $(x, \tan x)$ be any point on the graph then $(-x, -\tan x)$ will also be a point on the graph, i.e., the graph must be symmetrical in opposite quadrants.

The following table gives values of tan x corresponding to some typical values of x in $]-\pi/2, \pi/2[$.

x	$-\pi/3$	$-\pi/6$	0	$\pi/6$	$\pi/3$
tan x	−1·73	−·58	0	·58	1·73

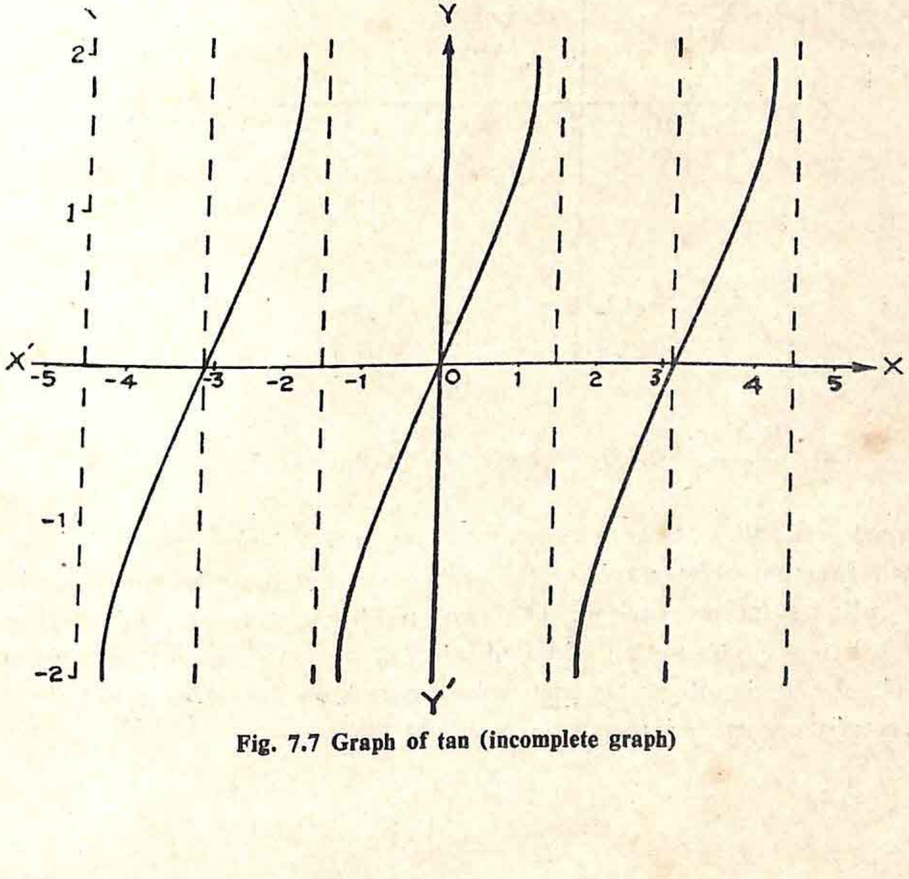

Fig. 7.7 Graph of tan (incomplete graph)

Since tan x is not defined when $x = \pm\pi/2$, therefore there will be no points on the graph corresponding to these values of x. (By periodicity, there will be no points on the graph corresponding to $x = k\pi \pm \pi/2$, k being any integer). A part of the graph is as shown in figure 7.7.

Throughout the entire domain the graph rises from the left to the right. This is just the graphical demonstration of the fact that tan x is strictly increasing on the interval $]\,k\pi - \pi/2, k\pi + \pi/2\,[$, k being any integer.

The graph of the cotangent function is similar to that of tangent and can be drawn in the same manner. A part of the graph is as shown in figure 7.8.

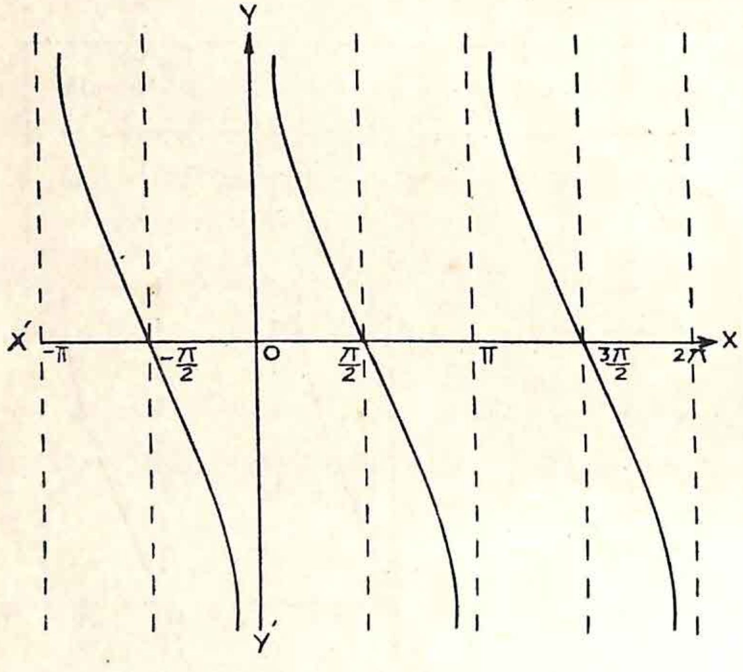

Fig. 7.8 Graph of cot (incomplete graph)

The secant and the cosecant functions. Figure 7.9 shows a part of the graph of the secant function. The graph of the cosine function has also been drawn on the same set of axes for comparison. The two graphs exhibit the relation $\sec x = 1/\cos x$ (whenever x is not an odd multiple of $\pi/2$). The ordinate of any point on the graph of the secant can be determined by taking the reciprocal of the ordinate of the corresponding point on the graph of the cosine.

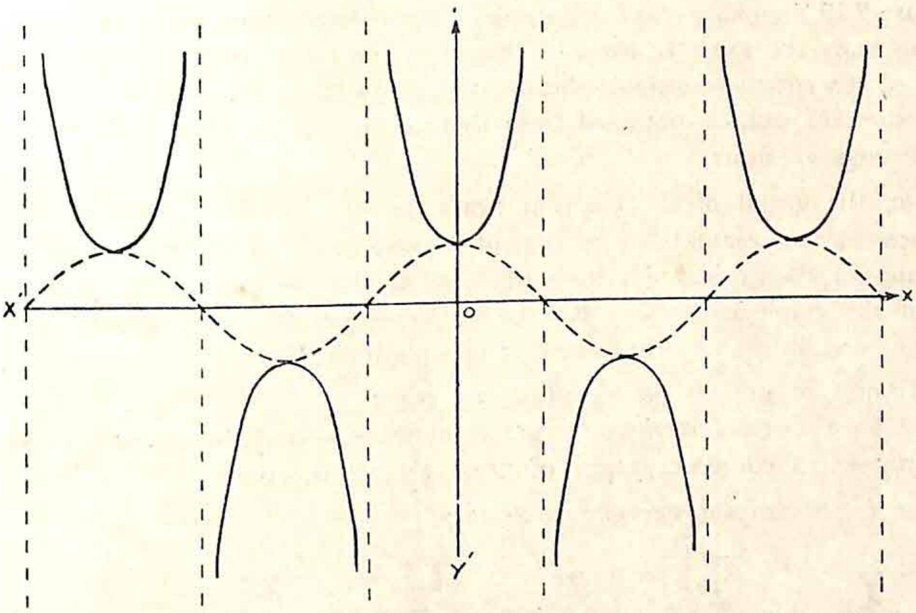

Fig. 7.9 Graphs of sec and cos (incomplete graphs) (sec——— cos — — — —)

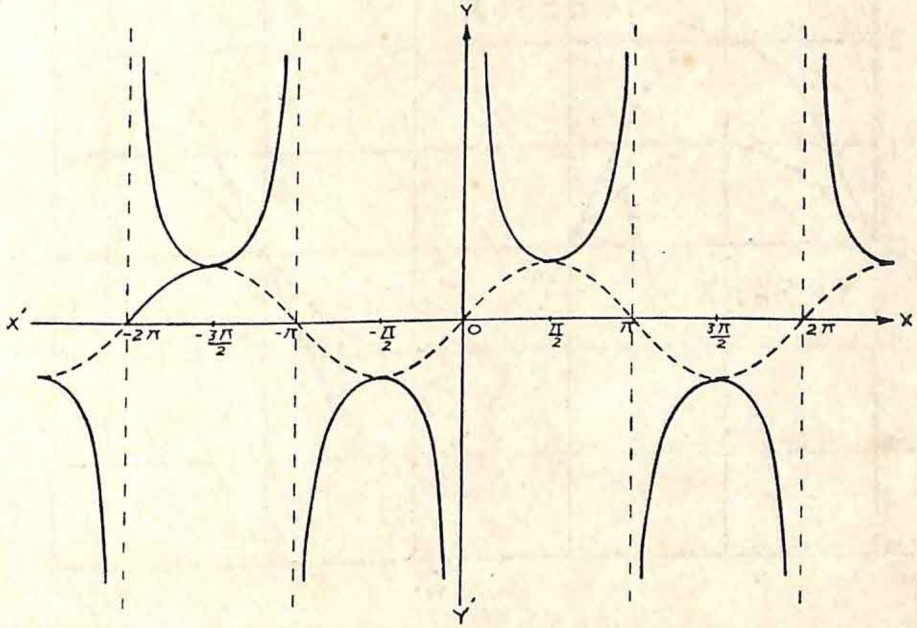

Fig. 7.10 Graphs of cosecant and sine (incomplete graphs)

$y = \sin x$ — — — — — —
$y = \csc x$ ————————

Figure 7.10 shows a part of the graph of the cosecant function. The graph of the cosecant bears the same relation to that of the secant as the graph of the sine bears to that of the cosine because of the identity csc $(\pi/2 + x) = \sec x$, i.e., the graph of the cosecant can be obtained from that of the secant by a translation to the right through $\pi/2$ units.

Also, the graph of the cosecant bears the same relation to that of the sine as the graph of the secant bears to that of the cosine, i.e., the ordinate of any point on the graph of the cosecant is the reciprocal of the ordinate of the corresponding point on the graph of the sine. For the sake of comparison, the graph of the sine has also been drawn on the same set of co-ordinate axes.

7.10. Graphs of $y = a \sin x$ and $y = a \cos x$

We have already drawn the graphs of the sine and the cosine functions. We now propose to consider graphs of more general functions.

Example 1. Sketch the curve $y = 2 \sin x$.

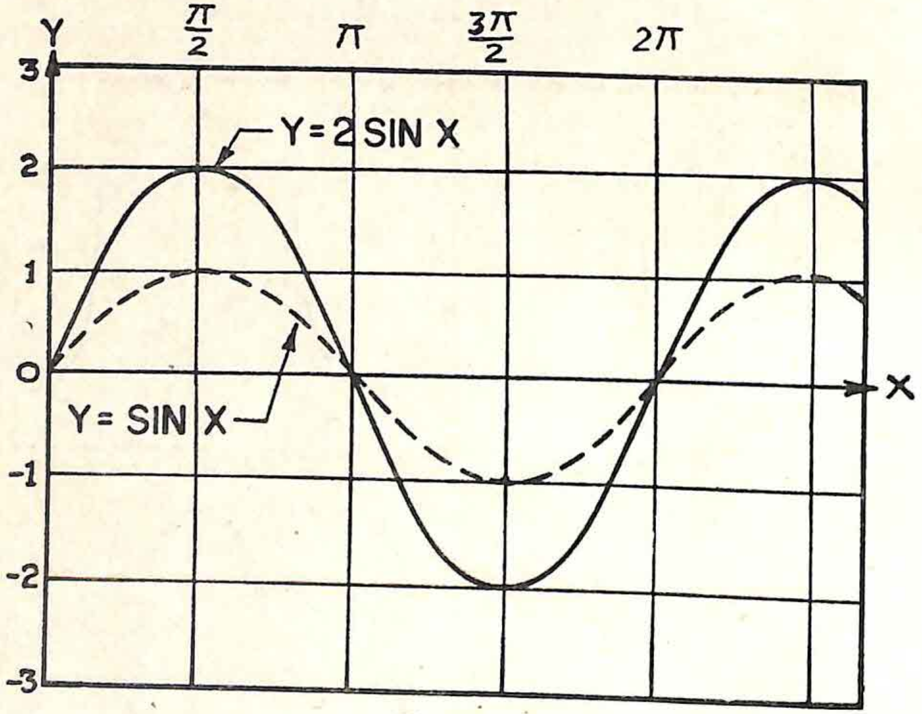

Fig. 7.11

$y = 2 \sin x$ ————

$y = \sin x$ – – – – – –

Solution. Since sin x varies between -1 and 1, therefore 2 sin x will vary between -2 and 2. The period of the function $\{(x, y) : y = 2 \sin x\}$ being the same as that of the sine function, is 2π. For any value of x, the value of y in $y = 2 \sin x$ is twice the value of y in $y = \sin x$. The graph of $y = 2 \sin x$ may therefore be obtained from that of $y = \sin x$ by stretching the graph by a factor 2 in the direction of the y-axis. The graph is as shown in figure 7.11. The graph of $y = \sin x$ has also been shown for comparison.

Example 2. Sketch the curve $y = 3 \cos x$.

Solution. The function $\{(x, y) : y = 3 \cos x\}$ is periodic with 2π as the period. The range of the function is $[-3, 3]$. The graph is obtained by stretching each ordinate of $y = \cos x$ by a multiple 3. A sketch is as shown in figure 7.12.

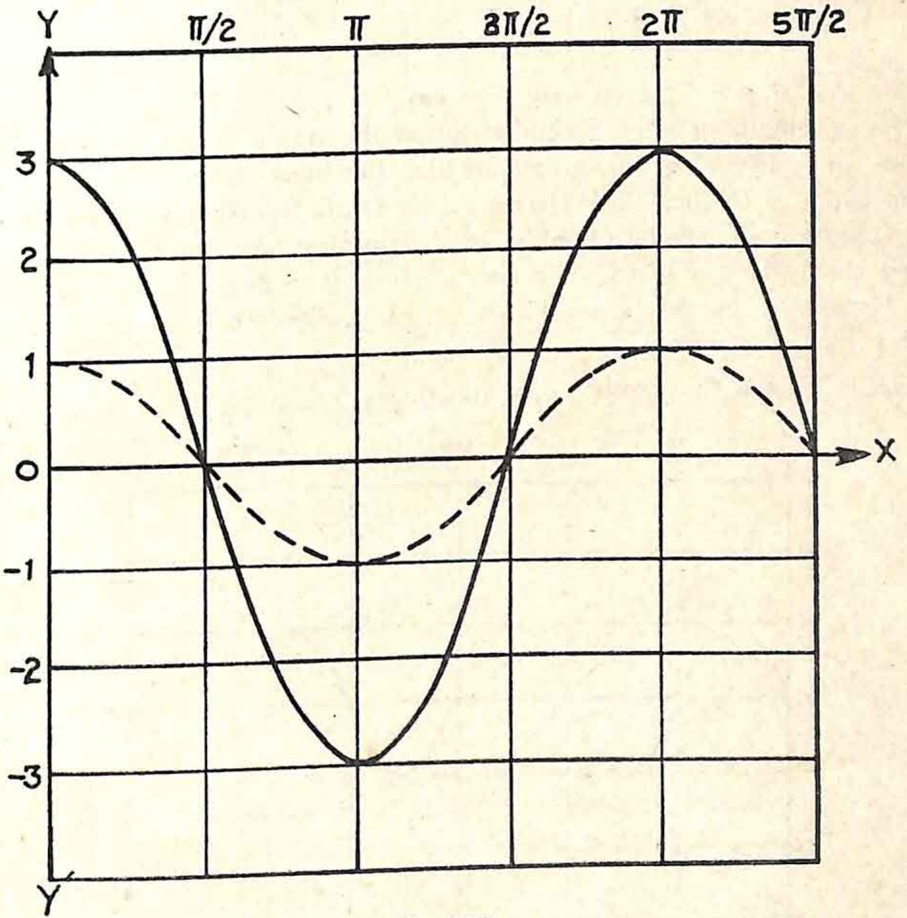

Fig. 7.12

$y = \cos x$ – – – – – –

$y = 3 \cos x$ ————

EXERCISE

1. Sketch the graphs of each pair of equations in the interval $[0, 2\pi]$:

(i) $y = \sin x$, $y = 1{\cdot}5 \sin x$.
(ii) $y = \sin x$, $y = 3 \sin x$.
(iii) $y = \sin x$, $y = 2{\cdot}5 \sin x$.
(iv) $y = \sin x$, $y = -\sin x$.
(v) $y = \sin x$, $y = -2 \sin x$.
(vi) $y = \cos x$, $y = 1{\cdot}5 \cos x$.
(vii) $y = \cos x$, $y = 2 \cos x$.
(viii) $y = \cos x$, $y = 2{\cdot}5 \cos x$.
(ix) $y = \cos x$, $y = -\cos x$.
(x) $y = \cos x$, $y = -2 \cos x$.

7.11. Graphs of $y = \sin bx$ and $y = \cos bx$.

We can obtain another generalization of the graphs of the sine and the cosine functions by considering the graphs of the functions defined by the equations $y = \sin bx$, $y = \cos bx$. The extreme values of the function $y = \sin bx$ are -1 and 1. The period may be obtained by determining how much x changes in producing a change of 2π in bx. As x changes from 0 to $2\pi/b$, bx changes from 0 to 2π b. Hence the period of sin bx is $2\pi/|b|$. Similarly, the period of cos bx is $2\pi/|b|$.

Example 3. Sketch the graph of the function $y = \sin 2x$.

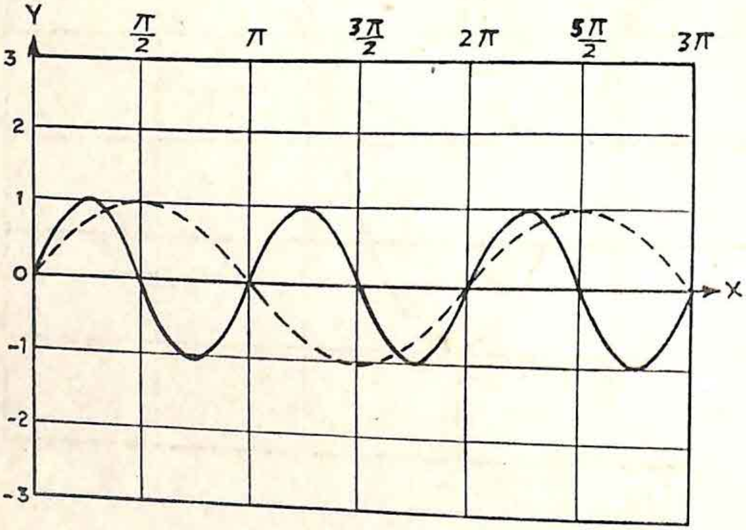

Fig. 7.13

$y = \sin x$ – – – – – –
$y = \sin 2x$ ————

Solution. The extreme values are -1 and 1. The period is $2\pi/2$, i.e., π. The graph is similar to that of sin x and is obtained from it by compressing it by a factor 1/2 along the x-axis. The graph is as shown in figure 7.13.

Example 4. Sketch a graph of the curve $y = \cos \frac{1}{2} x$.

Solution. The extreme values are -1 and 1. The period is $2\pi/\frac{1}{2}$, i.e., 4π. The graph is similar to that of the cosine function and is obtained from it by stretching it along the x-axis by a factor 2. The graph is as shown in figure 7.14.

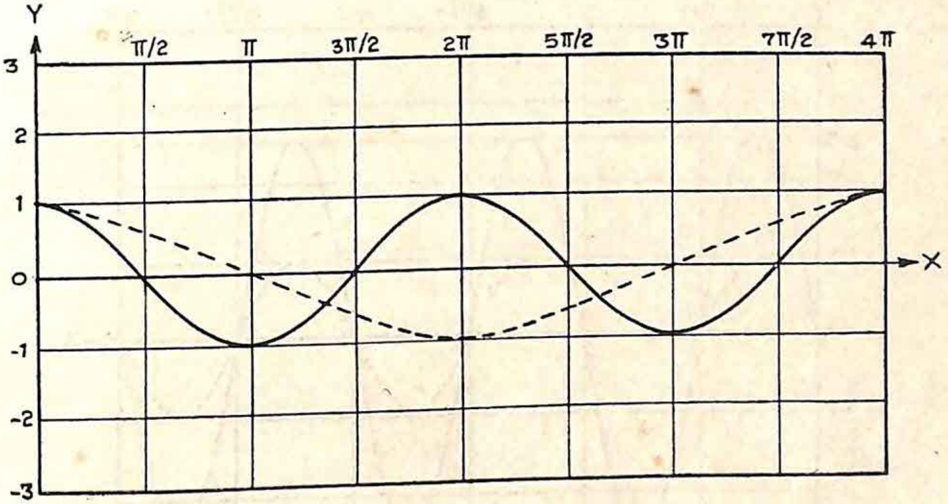

Fig. 7.14

$y = \cos x$ - - - - - - -

$y = \cos \frac{1}{2} x$ ————

EXERCISE

1. Sketch the graph of each of the following curves over one period. Along with each graph, draw also the graph of the sine function or the cosine function, as the case may be.

(i) $y = \sin 3x$. (ii) $y = \sin \frac{1}{2} x$.
(iii) $y = \sin 2x$. (iv) $y = \sin \frac{3}{2} x$.
(v) $y = \cos 2x$. (vi) $y = \cos 3x$.
(vii) $y = \cos \frac{3}{2} x$. (viii) $y = \cos \frac{5}{2} x$.

7.12. Graphs of y = a sin bx and y = a cos bx

The graph of $y = a \sin bx$ can be obtained from that of $y = \sin x$ by first drawing the graph of $y = \sin bx$ (as in example 3) and then stretching the graph of $y = \sin bx$ by a factor a along the y-axis (as in example 1). The extreme values of y

$= a \sin bx$ are $\pm a$, and the period is $2\pi/|b|$. The same is also true of the curve $y = \cos bx$.

The following example will illustrate the method.

Example 5. Sketch the curve $y = 2{\cdot}5 \sin 2x$.

Solution. We shall first draw the graph of $y = \sin 2x$, and then stretch the graph along the y-axis by a factor 2·5.

The graph is as shown in the figure 7.15

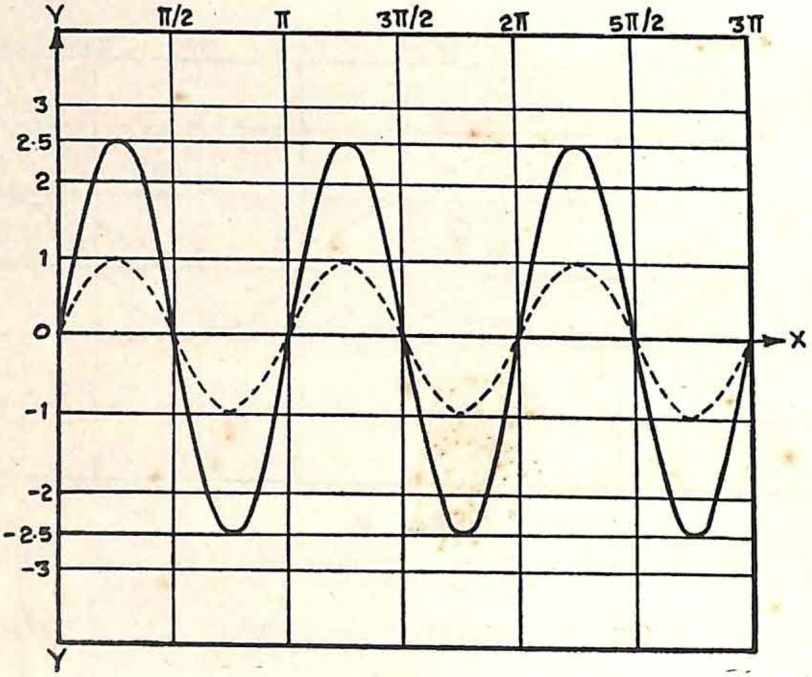

Fig. 7.15

$y = \sin 2x$ — — — — — —

$y = 2{\cdot}5 \sin 2x$————

EXERCISE

1. Sketch the graph of each of the following curves over one period :

(i) $y = 2 \sin 2x$.
(ii) $y = 1{\cdot}5 \sin 2x$.
(iii) $y = 2 \sin 3x$.
(iv) $y = 2{\cdot}5 \sin 3x$.
(v) $y = 1{\cdot}5 \sin \frac{1}{2}x$.
(vi) $y = 3 \sin \frac{1}{2}x$.
(vii) $y = 2 \cos 3x$.
(viii) $y = 2 \cos \frac{1}{2}x$.
(ix) $y = 1{\cdot}5 \cos 2x$.
(x) $y = 2{\cdot}5 \cos 2x$.
(xi) $y = 2{\cdot}5 \cos 3x$.
(xii) $y = 3 \cos 2x$.

7.13. Graphs of $y = \sin(x + c)$ and $y = \cos(x + c)$

The graph of $y = \sin(x + c)$ is similar to that of $y = \sin x$ and is obtained by translating it through a distance c units to the left. The extreme values are ± 1, and the period is 2π. The graph of $y = \cos(x + c)$ is similar to that of $y = \cos x$ and can be obtained from it by translating to the left through a distance c units. The extreme values are again ± 1 and the period is 2π.

Example 6. Sketch the graph of $y = \sin(x + \pi/6)$.

Solution. The graph of $y = \sin(x + \pi/6)$ is obtained by translating the graph of $y = \sin x$ to the left through a distance $\pi/6$ units. This is equivalent to shifting the origin (and therefore the y-axis also) to the right through a distance $\pi/6$ units. Therefore we shall first draw the graph of $y = \sin x$ and then shift the origin to the right through a distance $\pi/6$ units. The graph is as shown in figure 7.16.

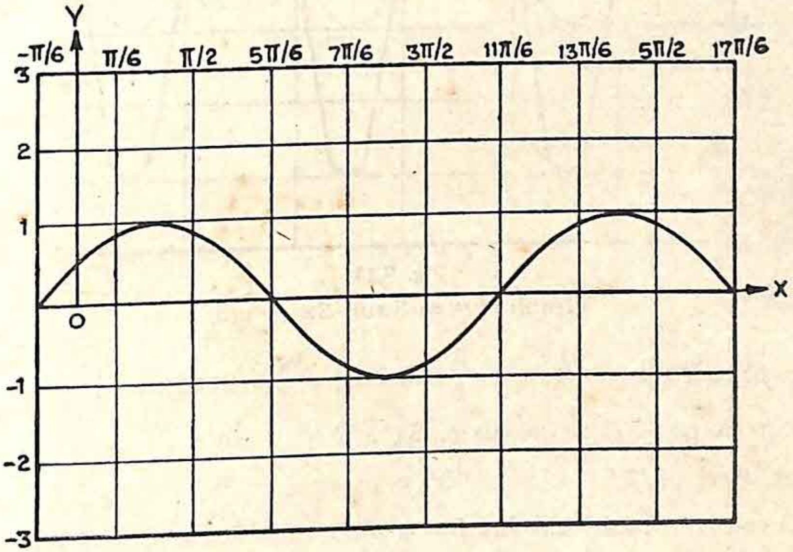

Fig. 7.16

Graph of $y = \sin(x + \pi/6)$

7.14. Graph of $y = a \sin(bx + c)$

The graph of $y = a \sin(bx + c)$ is similar to that of $y = a \sin bx$ and is obtained from it by translating it through a distance c/b units to the left (or equivalently, shifting the origin and the y-axis through a distance c/b units to the right). The following example will illustrate the method.

Example 7. Sketch the curve $y = 3 \sin(2x + \pi/3)$.

Solution. Since $3 \sin(2x + \pi/3) = 3 \sin\{2(x + \pi/6)\}$, therefore the graph is the same as that of $y = 3 \sin 2x$ with the origin shifted to the right through a distance $\pi/6$ units. The graph is as shown in the figure 7.17.

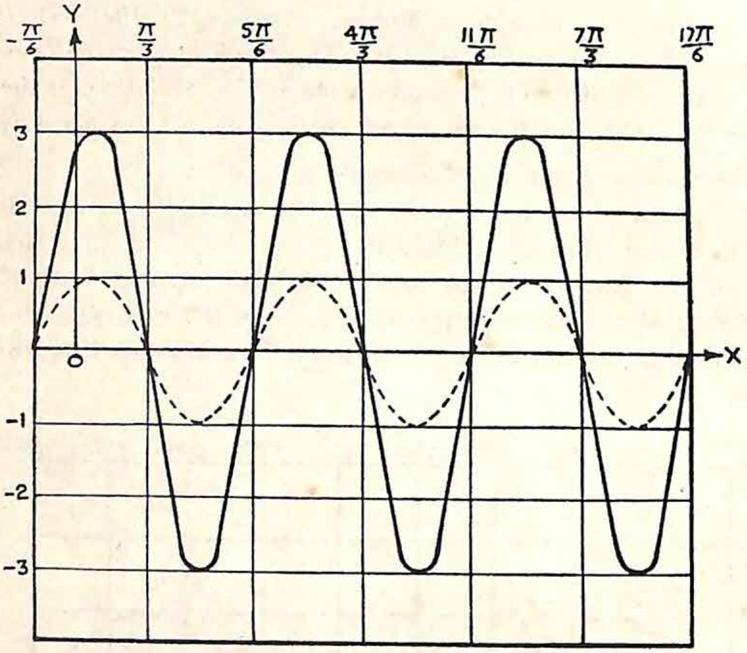

Fig. 7.17

Graph of $y = 3 \sin (2x + \pi/3)$

Example 8. Sketch the curve $y = \dfrac{3}{2} \sin 2x + \dfrac{3\sqrt{3}}{2} \cos 2x$.

Solution. Let us put $3/2 = a \cos c$, $3\sqrt{3}/2 = a \sin c$.

We then have $(3/2)^2 + (3\sqrt{3}/2)^2 = a^2$, or $a^2 = 9$.

If we choose the positive value for a, then $a = 3$, $\cos c = \frac{1}{2}$, $\sin c = \sqrt{3}/2$, so that we may take $c = \pi/3$. With this choice the given equation can be written as

$$\begin{aligned} y &= a \sin 2x \cos c + a \cos 2x \sin c, \\ &= a \sin (2x + c), \\ &= 3 \sin (2x + \pi/3). \end{aligned}$$

This equation being the same as in example 7, the desired graph is as shown in figure 7.17.

EXERCISES

1. Sketch the graph of each of the following curves :

(i) $y = \sin (x + \pi/4)$. (ii) $y = \sin (x - \pi/6)$.

(iii) $y = \cos(x + \pi/3)$. (iv) $y = \cos(x - \pi/4)$.
(v) $y = \sin(2x + \pi/4)$. (vi) $y = \sin(2x - \pi/3)$.
(vii) $y = \cos(2x + \pi/3)$. (viii) $y = \cos(3x - \pi/2)$.
(ix) $y = 1{\cdot}5 \cos(2x + \pi/3)$. (x) $y = 2{\cdot}5 \sin(2x + \pi/3)$.
(xi) $y = 3 \cos(2x - \pi/3)$. (xii) $y = 2 \sin(3x - 3\pi/4)$.
(xiii) $y = \cos x + \sqrt{3} \sin 2x$. (xiv) $y = 4 \cos 2x - 3 \sin 2x$.
(xv) $y = \sqrt{2}(\cos x + \sin x)$.

7.15. Sketching of Curves by Composition of Ordinates

If we draw the graphs of two functions defined by the equations $y = f(x)$ and $y=g(x\text{-})$ on the same set of axes (and to the same scale), then the graph of the function defined by the equation $y = f(x) + g(x)$ can be obtained by composition of ordinates. Corresponding to the points on the curves $y = f(x)$, $y = g(x)$ having the same abscissa x, we can find graphically the algebraic sum of $f(x)$ and $g(x)$ and thus get a point on the curve $y = f(x) + g(x)$. After plotting a few points in this manner we can join them by a smooth curve to get a sketch of $y = f(x) + g(x)$.

Example 9. Sketch the curve $y = \frac{1}{2} x + \sin x$ by composition of ordinates.

Solution. Sketch the curves $y = \frac{1}{2}x$ and $y = \sin x$ on the same set of axes.

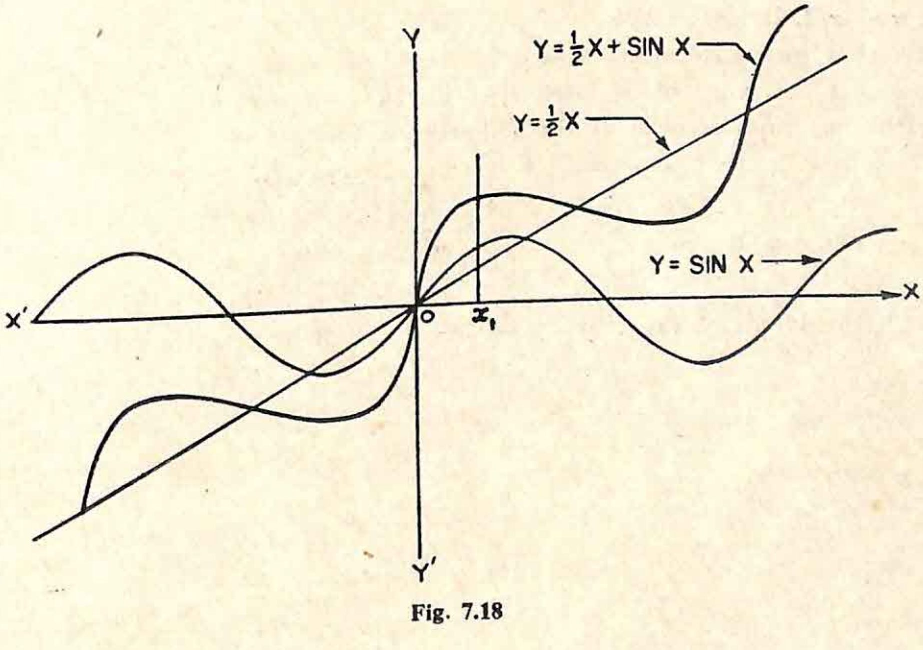

Fig. 7.18

Since the function given by $y = \frac{1}{2}x + \sin x$ is the sum of the functions given by $y = \frac{1}{2}x$ and $y = \sin x$, therefore the ordinate of its graph corresponding to any value x_1 of x is the algebraic sum of the ordinates of the graphs of $y = \frac{1}{2}x$ and $y = \sin x$ corresponding to the value $x = x_1$. The graph is as shown in figure 7.18.

EXERCISE

Sketch the graph of each of the following curves by composition of ordinates :

(i) $y = x + \sin x$.
(ii) $y = \frac{1}{3}x + \sin x$.
(iii) $y = 1 + 2 \cos 2x$.
(iv) $y = \sin x + \sin 2x$.
(v) $y = \sin x + 2 \sin 2x$.
(vi) $y = \cos x + \cos 2x$.
(vii) $y = 2 \cos x + \cos 2x$.
(viii) $y = \cos x + \sin 2x$.

ASSORTED EXERCISES ON CHAPTER 7

1. Show, by means of examples, that given two functions f and g, it may happen that

   (i) neither $g \circ f$ exists nor $f \circ g$ exists ;
   (ii) $g \circ f$ exists but $f \circ g$ does not exist ;
   (iii) $f \circ g$ exists but $g \circ f$ does not exist ;
   (iv) $f \circ g$ and $g \circ f$ both exist but are not equal ;
   (v) $f \circ g$ and $g \circ f$ both exist and are equal.

2. Show that for each integer k, sin x strictly increases from -1 to 1 on the interval $[2k\pi - \pi/2, 2k\pi + \pi/2]$.
3. Show that for each integer k, cot is strictly decreasing on $]k\pi, (k+1)\pi[$.
4. Show that the range of csc is $\mathbf{R} \backsim \,]-1, 1[$.
5. Sketch the graph of each of the following curves :

   (i) $y = 1{\cdot}5 \cos x$.
   (ii) $y = 2{\cdot}5 \sin x$.
   (iii) $y = \sin 2x$.
   (iv) $y = 2{\cdot}5 \sin 3x$.
   (v) $y = 2 \cos 2x$.
   (vi) $y = \sin (x + \pi/3)$.
   (vii) $y = 1{\cdot}5 \sin (2x + \pi/3)$.
   (viii) $y = 3 \sin x + 4 \cos x$.

6. Sketch the graph of the curve $y = 2x + 3 \sin x$ by composition of ordinates.

Chapter 8

Inverse Trigonometric Functions

8.1. Inverse Functions

Let f be a function with domain X and range Y. If we can find a function g with domain Y and range X such that the composite function g o f is the identity function on X, then we say that the function g is an inverse of f.

Now, by saying that g o f is the identity function on X we mean that (g o f) (x) = x, for all $x \in X$, i.e. $g[f(x)] = x$, for all $x \in X$. Therefore we may also state the idea expressed in the preceding paragraph as follows :

A function g with domain Y and range X is said to be an inverse of a function f with domain X and range Y if $g\,[f(x)] = x$ for all $x \in X$.

Example 1. Let f be the function with domain **R** and range **R** defined by the equation

$$f(x) = 3\,x + 2 \text{ for all } x \in \mathbf{R}.$$

Also, let g be the function with domain **R** and range **R** defined by the equation

$$g(x) = \tfrac{1}{3}\,(x-2) \text{ for all } x \in \mathbf{R}.$$

Then for each $x \in \mathbf{R}$,

$$\begin{aligned} g\,(f(x)) &= g\,(3x + 2), \\ &= \tfrac{1}{3}\,[(3x + 2) - 2], \\ &= x. \end{aligned}$$

The function g is, therefore, an inverse of f.

Example 2. Let f be the function with domain $\mathbf{R} \sim \{0\}$ and range $\mathbf{R} \sim \{0\}$ defined by the equation

$$f(x) = 1/x, \text{ for all } x \in \mathbf{R} \sim \{0\}.$$

For each $x \in \mathbf{R} \sim \{0\}$,

$$\begin{aligned} f(f(x)) &= f\,(1/x), \\ &= \frac{1}{1/x}, \\ &= x, \end{aligned}$$

showing that the function f is an inverse of itself.

Example 3. Let f be the function with domain $\mathbf{R}^+ \cup \{0\}$ and range $\mathbf{R}^+ \cup \{0\}$ defined by the equation

$$f(x) = \sqrt{x}, \text{ for all } x \in \mathbf{R}^+ \cup \{0\},$$

where $\sqrt{x}$ denotes the positive square root of x.

Also, let g be the function with domain $\mathbf{R}^+ \cup \{0\}$ and range $\mathbf{R}^+ \cup \{0\}$ defined by the equation

$$g(x) = x^2, \text{ for all } x \in \mathbf{R}^+ \cup \{0\}.$$

For each $x \in \mathbf{R}^+ \cup \{0\}$,

$$\begin{aligned} g(f(x)) &= g(\sqrt{x}), \\ &= (\sqrt{x})^2, \\ &= x, \end{aligned}$$

showing that the function g is an inverse of f.

The following theorem shows that a function cannot have more than one inverse.

Theorem 1. Let f be a function with domain X and range Y, and let g and h be inverses of f. Then $g = h$.

Proof. Since g and h are inverses of f, therefore each of them is a function with domain Y and range X. To prove that $g = h$, it is enough to show that

$$g(y) = h(y) \text{ for all } y \in Y.$$

Let y be any element of Y. Since f is a function with domain X and range Y, therefore for some x in X, we must have

$$f(x) = y.$$

Now $g(y) = g(f(x)) = x$, since g is an inverse of f.
Similarly $h(y) = h(f(x)) = x$.

The above equation shows that $g(y) = h(y)$. Thus we find that for each $y \in Y$,

$$g(y) = h(y).$$

Hence $g = h$.

Remark. In view of theorem 1 we can talk of *the* inverse of a function rather than *an* inverse.

We shall now show that if a function g be the inverse of a function f, then f is the inverse of g.

Theorem 2. Let f be a function with domain X and range Y and let g be the inverse of f. Then f is the inverse of g.

Proof. Since g is the inverse of f, therefore g is a function with domain Y and range X.

Now g is a function with domain Y and range X, and f is a function with domain X and range Y. To prove that f is the inverse of g, it is enough to show that

$$f(g(y)) = y \text{ for all } y \in Y.$$

Let y be any element of Y. Since f is a function with domain X and range Y, therefore we can choose an $x \in X$, such that $f(x) = y$.

$$\begin{aligned} \text{Then} \quad f(g(y)) &= f(g(f(x))), \\ &= f(x), \text{ since } g(f(x)) = x \text{ for all } x \in X, \\ &= y. \end{aligned}$$

Hence f is the inverse of g.

In view of theorem 2, we can talk of a pair of functions as *inverse functions* rather than talking of one of them as being the inverse of the other. The functions f and g in each of the examples 1-3 above are inverse functions.

As the following examples will show, it is not necessary for a function to have an inverse.

Theorem 3. Let f and g be inverse functions such that the domain of f (range of g) is X and the range of f (domain of g) is Y. Then f and g are both one-to-one.

Proof. To prove that f is one-to-one amounts to showing that for each pair x_1, x_2 of distinct elements of X, $f(x_1)$ and $f(x_2)$ are distinct elements of Y.

Let x_1 and x_2 be elements of X such that $x_1 \neq x_2$. Since g is a function with domain Y and range X, therefore we can choose y_1 and y_2 in Y such that $g(y_1) = x_1$, $g(y_2) = x_2$. Since $x_1 \neq x_2$, therefore, by the definition of a function, $y_1 \neq y_2$.

$$\begin{aligned} \text{Now} \quad f(x_1) &= f(g(y_1)) = y_1, \\ f(x_2) &= f(g(y_2)) = y_2, \end{aligned}$$

since f and g are inverse functions.

From the above equalities we find that $f(x_1) \neq f(x_2)$. Hence f is one-to-one. Similarly g is also one-to-one.

The above theorem shows that every function need not have an inverse. A function which possesses an inverse is said to be invertible (or inversible). If a function is invertible, by theorem 3, it must be one-to-one. This necessary condition for invertibility gives us a method of finding examples of functions which do not possess inverses. If a function is *not* one-to-one, then it cannot have an inverse. None of the following functions has an inverse :

1. Let f be a function with domain **R** and range $\mathbf{R}^+ \cup \{0\}$ defined by the equation

$$f(x) = |x|, \text{ for all } x \in \mathbf{R}.$$

f is not one-to-one because

$$f(-2) = f(2) = 2.$$

2. Let f be a function with domain **R** and range $\mathbf{R}^+ \cup \{0\}$ defined by the equation

$$f(x) = x^2, \text{ for all } x \in \mathbf{R}.$$

f is not one-to-one because

$$f(-2) = f(2) = 4.$$

A very natural question now arises—Given a function f, when can we be sure that it is invertible ?

The following theorem gives an answer to the question posed above.

Theorem 4. Every one-to-one function is invertible.

Proof. Let f be a one-to-one function with domain X and range Y. We shall construct a function g with domain Y and range X such that $g(f(x)) = x$ for all $x \in X$.

Let y be any member of Y. Since y belongs to the range of f, therefore there is an element x in domain of f such that $f(x) = y$. Also since f is one-to-one, therefore for each $y \in Y$ there is exactly one $x \in X$ such that $f(x) = y$. Let us set $g(y) = x$. Then g is a function with domain Y.

Let now x be any member of X. If we denote f(x) by y, then, by the definition of g, we have $g(y) = x$, i.e., $g(f(x)) = x$. This shows that the range of g is X and that g is the inverse of f.

This completes the proof of the theorem.

We shall now obtain another criterion for a function to be invertible.

Theorem 5. A function f with domain X is invertible if and only if the relation $\{(f(x), x) : x \in X\}$ is a function. Also, the inverse of f is then $\{(f(x), x) : x \in X\}$.

Proof. f is invertible $\Longleftrightarrow$ f is one-to-one,

$$\Longleftrightarrow x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2),$$

$$\Longleftrightarrow \{(f(x), x) : x \in X\} \text{ is a function,}$$

Let us now assume that f is invertible and let $g = \{(f(x), x) : x \in X\}$.

Then domain of $g = \{f(x) : x \in X\}$,

$= \text{range of } f.$

Range of $g = \{x : x \in X\}$,

$= X,$

$= \text{domain of } f.$

Also, if x be any member of X, then $(f(x), x) \in g$, i.e., $g(f(x)) = x$.

Therefore g is a function whose domain is the range of f, whose range is the domain of f, and which is such that $g(f(x)) = x$ for all $x \in X$.

Hence g is the inverse of f.

The above theorem gives a convenient method of finding whether a given function is invertible, as also of finding the inverse of an invertible function.

For any function f with domain X, let us call $\{(f(x), x) : x \in X\}$ the *inverse relation* of f. The above theorem can then be stated in the following form :

A function f is invertible if and only if its inverse relation is a function. Also, if this be the case, then this function is the inverse of f.

8.2. Inverse Trigonometric Functions

We shall now consider the inverse relations of the trigonometric functions. We shall see that while the trigonometric functions are not invertible, it is possible to define certain functions, to be called inverse circular functions, which are subsets of the inverse relations of the trigonometric functions.

Let us first of all consider the sine function. We know that

$$\sin = \{(x,y) : y = \sin x, x \in \mathbf{R}\}.$$

The inverse relation of the sine is the relation

$$\{(y,x) : y = \sin x, x \in \mathbf{R}\},$$

$$\text{i.e., } \{(x,y) : x = \sin y, y \in \mathbf{R}\}.$$

This relation is called the *inverse sine relation or arcsine relation* and is written as $\sin^{-1}$ or arcsin.

Thus

$$\arcsin = \{(x,y) : x = \sin y, y \in \mathbf{R}\} \qquad \text{............(1)}$$

The arcsin relation is *not* a function because for a given value of x there may exist distinct numbers y_1, y_2 such that (x, y_1) and (x, y_2) both belong to arcsin. For example, if we consider $x = 0$, we find that . . . . $(0, -\pi)$, $(0,0)$, $(0, \pi)$, $(0, 2\pi)$, $(0, 3\pi)$, ... all belong to arcsin, i.e., $(0, n\pi) \in$ arcsin for each $n \in \mathbf{Z}$. We can however obtain a function from the arcsin relation by placing a suitable restriction on the range of the relation (1) so that for each x in the domain of the relation there exists only one member of the range. Let us see how it can be done.

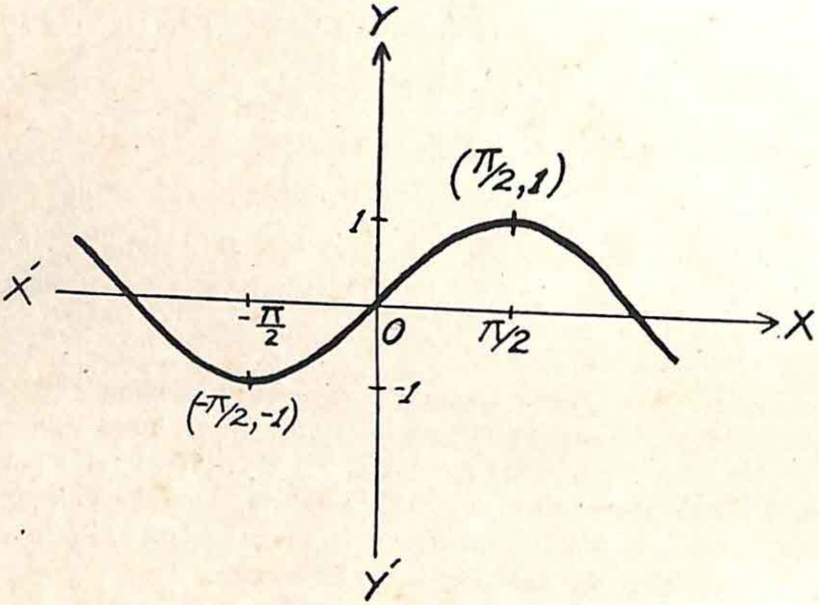

Fig. 8.1 Graph of the sine function (incomplete graph)

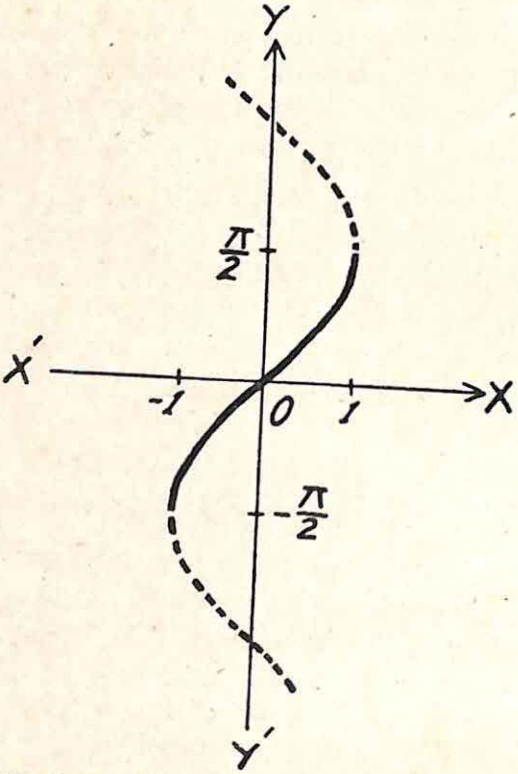

Fig. 8.2 Graph of the arcsine relation (incomplete graph)

Let us look at the graph of the sine function (Fig. 8.1) $\{(x,y) : y = \sin x, x \in \mathbf{R}\}$. For each real number a such that $-1 \leqslant a \leqslant 1$, there is one and only one real number x such that $\sin x = a$ and $-\pi/2 \leqslant x \leqslant \pi/2$. We denote this real number by arcsin a or $\sin^{-1} a$. Thus $x = \sin^{-1} a$ if and only if $\sin x = a$, and $-\pi/2 \leqslant x \leqslant \pi/2$. For example, $\sin^{-1} \frac{1}{2} = \pi/6$, $\sin^{-1}(-\sqrt{3}/2) = -\pi/3$. The set of all ordered pairs (x, arcsin x) where $-1 \leqslant x \leqslant 1$ is called the *arcsine function* or *inverse sine function* and is written as arcsin or $\sin^{-1}$.

Thus

$$\begin{aligned}\arcsin &= \{(x, \arcsin x) : -1 \leqslant x \leqslant 1\},\\ &= \{(x, y) : y = \arcsin x, -1 \leqslant x \leqslant 1\}.\end{aligned}$$

It can be easily seen that arcsin is in fact a function. (To see this amounts to verifying that $\arcsin x_1 \neq \arcsin x_2 \Rightarrow x_1 \neq x_2$; and this is true because the sine function is strictly increasing in $[-\pi/2, \pi/2]$, so that if we write $y_1 = \arcsin x_1$, $y_2 = \arcsin x_2$, then $x_1 = \sin y_1$, $x_2 = \sin y_2$ and $y_1 \neq y_2 \Rightarrow \sin y_1 \neq \sin y_2 \Rightarrow x_1 \neq x_2$.)

Figure 8.3 shows the graph of the arcsine function.

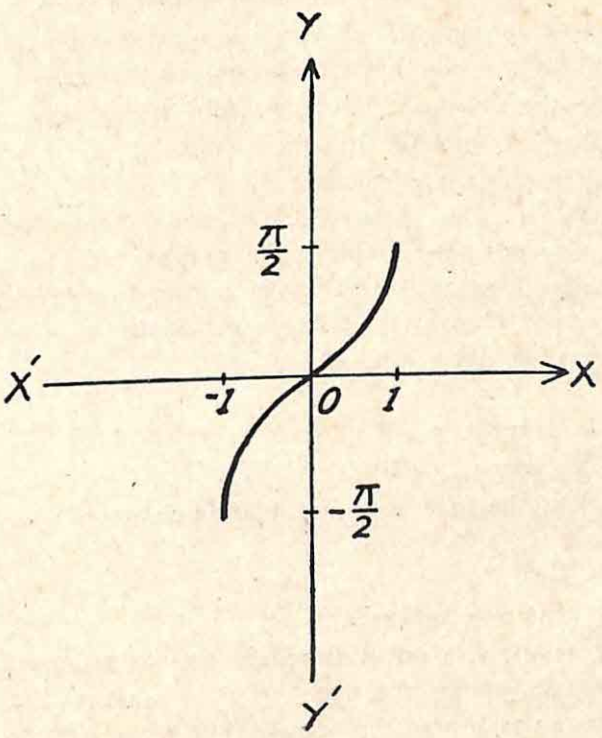

Fig. 8.3 Graph of the arcsine function

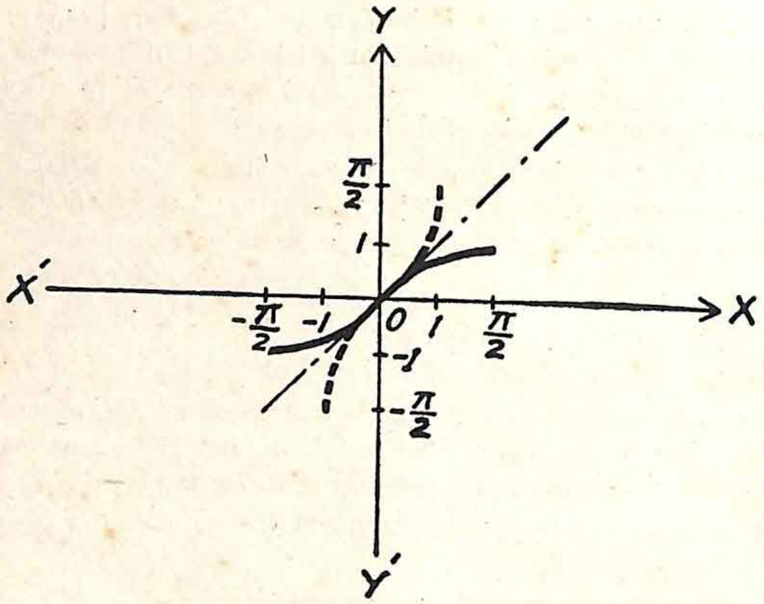

Fig. 8.4 Graphs of the sine and arcsine functions

Figure 8.4 shows the graphs of the sine function (only that part for which $-\pi/2 \leqslant x \leqslant \pi/2$) and arcsine function. It can be seen that each of them is the reflection of the other in the line $y=x$, i.e., if we fold the paper along the line $y=x$, then the two graphs will coincide. In other words, if (x, y) is a point on one of them, then (y, x) is a point on the other.

Let us once again consider figure 8.3. It shows the graph of the arcsine relation. The undotted part of the graph is the graph of the arcsine function. This shows that the arcsine function is a subset of the arcsin relation.

The arccosine function. The inverse relation of the function

$$\cos = \{(x,y) : y = \cos x, x \in \mathbf{R}\}$$

is the relation

$$\{(y,x) : y = \cos x, x \in \mathbf{R}\},$$

$$\text{i.e., } \{(x,y) : x = \cos y, y \in \mathbf{R}\}.$$

This relation is called the *inverse cosine relation* or *arccosine relation* and is written as $\cos^{-1}$ or arccos.

Thus

$$\text{arccos} = \{(x,y) : x = \cos y, y \in \mathbf{R}\}.$$

The arccosine relation is not a function (why ?), but we can obtain a function from the arccosine relation by placing a suitable restriction on the range of the relation (2) so that for each x in the domain of the relation there exists only one member of the range. Let us see as how this can be done.

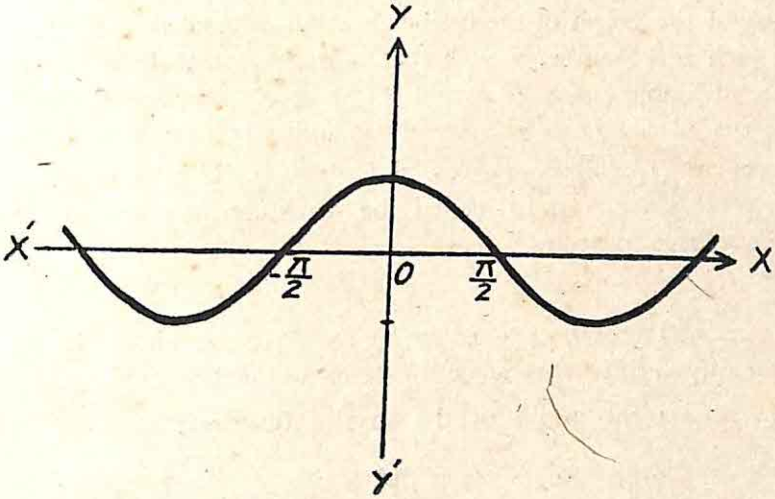

Fig. 8.5 Graph of the cosine function (incomplete graph)

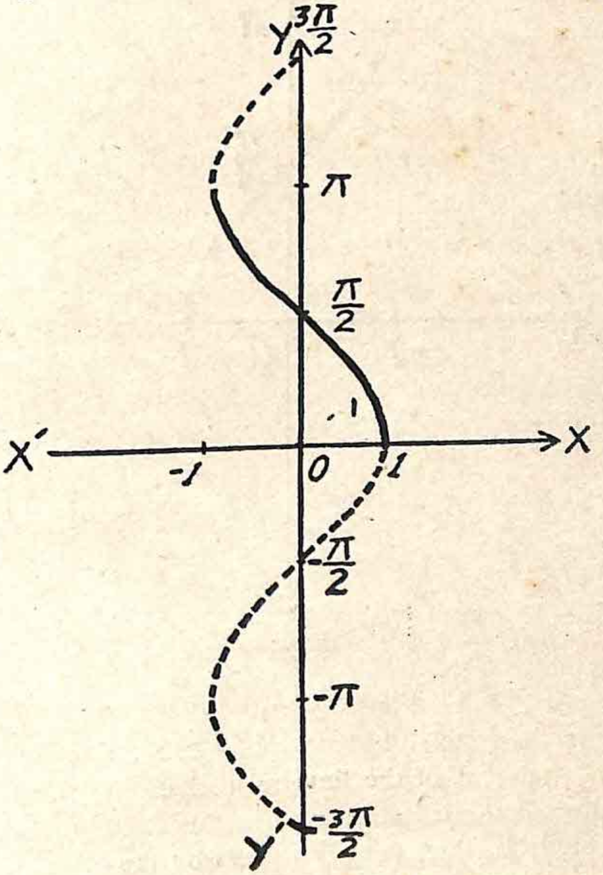

Fig. 8.6 Graph of the arccosine relation (incomplete graph)

Let us look at the graph of the cosine function (Fig. 8.5) $\{(x,y) : y = \cos x, x \in \mathbf{R}\}$. For each real number a such that $-1 \leqslant a \leqslant 1$ there is one and only one real number x such that $\cos x = a$ and $0 \leqslant x \leqslant \pi$. We denote this real number by arccos a or $\cos^{-1} a$. Thus $x = \cos^{-1} a$ if and only if $\cos x = a$ and $0 \leqslant x \leqslant \pi$ For example, $\cos^{-1}(1/\sqrt{2}) = \pi/4$, $\cos^{-1}(-1) = \pi$. The set of all ordered pairs (x, arccos x), where $-1 \leqslant x \leqslant 1$ is called the *arccosine function* or *inverse cosine function* and is written as arccos or $\cos^{-1}$.

Thus

$$\arccos = \{(x, \arccos x) : -1 \leqslant x \leqslant 1\}.$$

It can be easily verified that arccosine is in fact a function. (How ?)

Figure 8.7 shows the graph of the arccsine function.

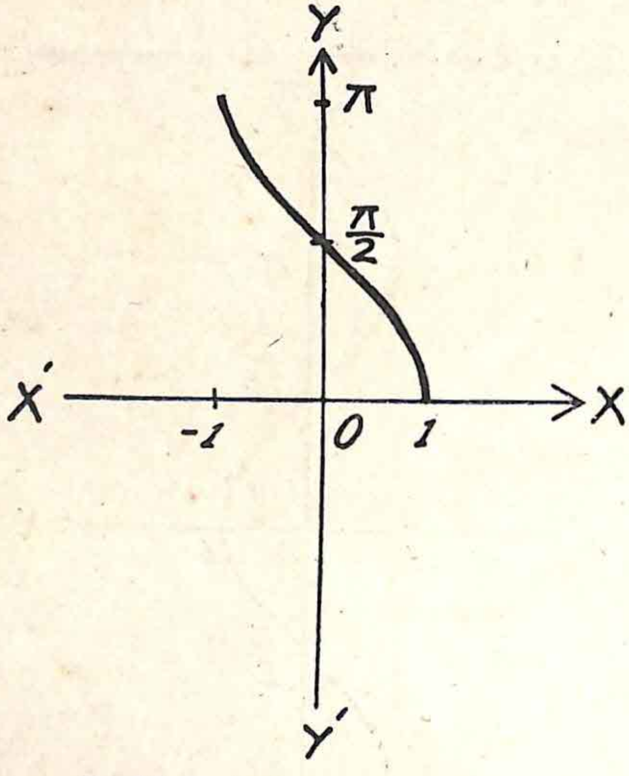

Fig. 8.7 Graph of the arccosine function

Figure 8.8 shows the graph of that part of the cosine function which corresponds to $0 \leqslant x \leqslant \pi$ and that of the 'inverse cosine' function. It can be easily seen that each of them is the reflection of the other in the line $y = x$, i.e., if (x,y) is a piont on one of them, then (y,x) is a point on the other.

Figure 8.6 shows the graph of the arccosine relation.

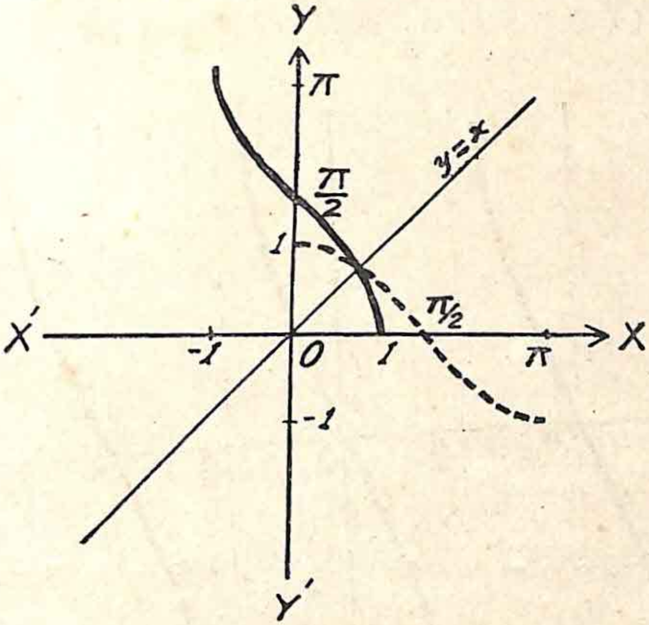

Fig. 8.8 Graph of the cosine and arccosine functions (incomplete graph)

The undotted part of the graph is the graph of the arccosine function. The graph shows that the arccosine function is a subset of the arccosine relation.

The arctangent function. The inverse relation of the function

$$\tan = \{(x,y) : y = \tan x, x \in \mathbf{R}^*\},$$

where $\mathbf{R}^*$, the domain of the tangent function, is the set $\mathbf{R} \sim \{(2n+1)\,\pi/2 : n \in \mathbf{Z}\}$, is the relation

$$\{(y, x) : y = \tan x, x \in \mathbf{R}^*\},$$

$$\text{i.e., } \{(x, y) : x = \tan y, y \in \mathbf{R}^*\}.$$

This relation is called the *inverse tangent relation* or *arctangent relation* and is written as $\tan^{-1}$ or arctan.

Thus

$$\arctan = \{(x, y) : x = \tan y, y \in \mathbf{R}^*\}. \qquad \text{......(3)}$$

The arctan relation is not a function (why ?), but we can obtain a function from the arctangent relation by placing a suitable restriction on the range of the relation (3) so that for each x in the domain of the relation there exists only one member of the range. Let us see how this can be done.

Let us look at the graph of the tangent function (Fig. 8.9) $\{(x,y) : y = \tan x, x \in \mathbf{R}\}$. The graph indicates that for each real number a, there is a unique real number x such that $\tan x = a$, $-\pi/2 \leqslant x \leqslant \pi/2$. We denote this number by arctan a or $\tan^{-1} a$. Thus $x = \tan^{-1} a$ if and only if $\tan x = a$ and x. For example, $\tan^{-1}(\sqrt{3}) = \pi/3$, $\tan^{-1}(-1/\sqrt{3}) = -\pi/6$.

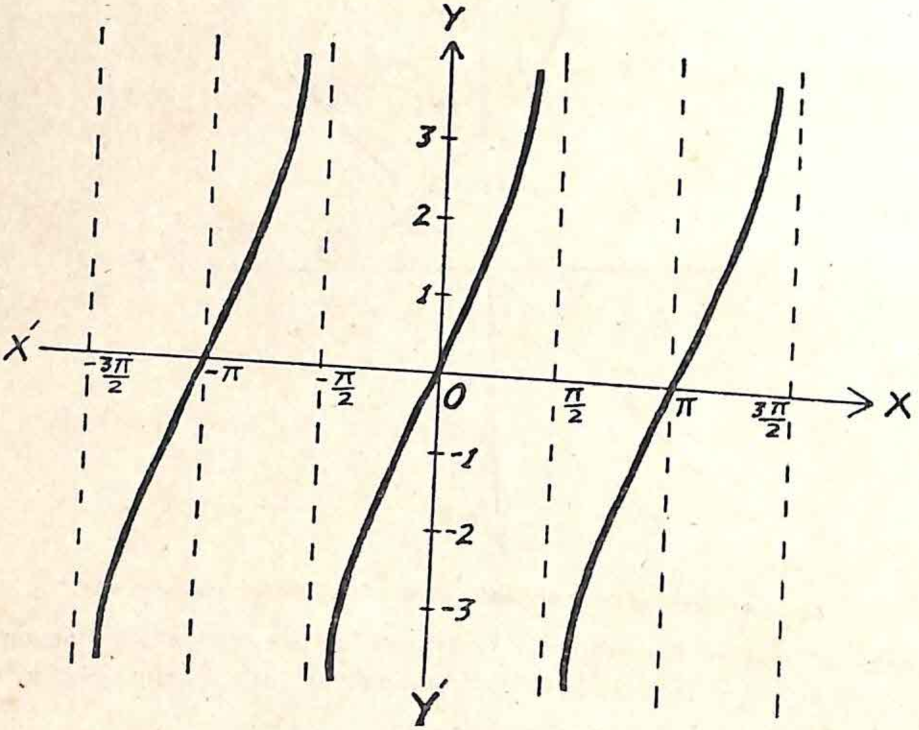

Fig. 8.9 Graph of the tangent function (incomplete graph)

The set of all ordered pairs (x, arctan x) where $-\pi/2 < x < \pi/2$ is called the *arctangent function* or *inverse tangent function* and is written as arctan or $\tan^{-1}$. Thus

$$\arctan = \{(x, \arctan x) : x \in \mathbf{R}\}.$$

It can be easily verified that arctan is in fact a function. (How ?)

Figure 8.10 shows the graph of the arctangent relation. The undotted part of the graph is the graph of the arctangent function.

The arccotangent, arcsecant and arccosecant functions. For each real number x_1 let the real number y be such that $\cot y = x$, $0 < y < \pi$ be denoted by arccot x. Then $\{(x, \text{arccot } x) : x \in \mathbf{R}\}$ is a function, called the *arccotangent* or *inverse co-tangent function* and is denoted by arccot, $\cot^{-1}$.

Thus $\text{arccot} = \{(x, \text{arccot } x) : x \in \mathbf{R}\}$. For each real number x such that

$|x| \geq 1$, we define arcsec x (also written as $\sec^{-1} x$) and arccsc x (also written as $\csc^{-1} x$) by setting

$$\text{arcsec } x = \arccos 1/x,$$
$$\text{arccsc } x = \arcsin 1/x.$$

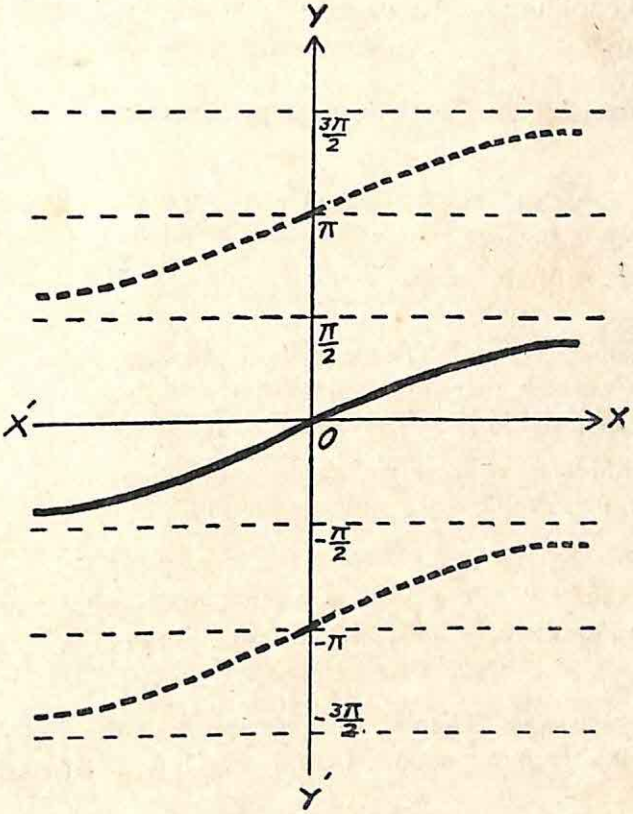

Fig. 8.10 Graph of the arctangent relation (incomplete graph)

The relations

$$\{(x, \text{arcsec } x) : x \in \mathbf{R}, |x| \geq 1\},$$
$$\{(x, \text{arccsc } x) : x \in \mathbf{R}, |x| \geq 1\},$$

are both functions. They are called the *arcsecant* (or inverse secant) and *arccosecant* (or inverse cosecant) functions respectively.

Thus,

$$\text{arcsec} = \{(x, \text{arcsec } x) : x \in \mathbf{R}, |x| \geq 1\},$$
$$\text{arccsc} = \{(x, \text{arccsc } x) : x \in \mathbf{R}, |x| \geq 1\}.$$

The six functions $\sin^{-1}$, ..., $\csc^{-1}$ defined above are called the *inverse trigonometric functions* (or *inverse circular functions*). Of these, only the first three are more important (because the last three can be defined in the terms of the first three ; in fact sec $^{-1}$, csc $^{-1}$ have already been defined in terms of $\cos^{-1}$ and $\sin^{-1}$ respectively and $\cot^{-1}$ can be expressed in terms of $\tan^{-1}$ because of the identity $\tan^{-1} x + \cot^{-1} x = \pi/2$, which can be easily verified).

For the sake of uniformity, let us agree that we shall henceforth use $\sin^{-1}x$, ..., $\csc^{-1}x$ for arcsin x, ..., arccsc x respectively.

8.3. Operations Involving Inverse Trigonometric Functions

Theorem 6.

(α) If $0 \leqslant x \leqslant 1$, then $0 \leqslant \sin^{-1} x \leqslant \pi/2$, $0 \leqslant \cos^{-1} x \leqslant \pi/2$.

(β) If $x \geq 0$, then $0 \leqslant \tan^{-1} x < \pi/2$, $0 < \cot^{-1} x \leqslant \pi/2$.

(γ) If $x \geq 1$, then $0 \leqslant \sec^{-1} x < \pi/2$, $0 < \csc^{-1} x \leqslant \pi/2$.

Proof.

(α) Let $\sin^{-1} x = u$. Then $x = \sin u$ and $-\pi/2 \leqslant u \leqslant \pi/2$. Since $x \leqslant 0$, therefore u cannot lie in $[-\pi/2, 0[$. Hence u lies in $[0, \pi/2]$, i.e., $0 \leqslant \sin^{-1} x \leqslant \pi/2$.

Again, let $\cos^{-1} x = v$. Then $x = \cos v$ and $0 \leqslant v \leqslant \pi$. Since $x \leqslant 0$, therefore v cannot lie in $]\pi/2, \pi]$. Hence v lies in $[0, \pi/2]$ i.e, $0 \leqslant \cos^{-1} x \leqslant \pi/2$.

(β) Let $\tan^{-1} x = u$. Then $x = \tan u$ and $-\pi/2 < u < \pi/2$. Since $x \geq 0$, therefore u cannot lie in $]-\pi/2, 0[$. Hence u lies in $[0, \pi/2[$, i.e., $0 \leqslant \tan^{-1} x < \pi/2$.

Agnin, let $\cot^{-1} x = v$. Then $x = \cot v$ and u lies either in $[-\pi/2, 0[$ or in $]0, \pi/2]$. Since $x \leqslant 0$, therefore v cannot liein $[-\pi/2, 0[$. Therefore v lies in $]0, \pi/2]$.

(γ) The proof is left as an exercise for the reader.

Remark. The above theorem shows that if $0 < x < 1$ then both $\sin^{-1} x$ and $\cos^{-1} x$ lie in $]0, \pi/2[$.

If we recall that for any real number y, the above value of sin y (resp. cos y, tan y etc.) is the sine (resp. cosine, tangent etc.) of the angle whose measure in radians is y, then in view of the above theorem we can compute sin ($\cos^{-1}x$), cos ($\tan^{-1}x$), etc. with the aid of right-angled triangles whenever x is positive. The following example will illustrate the point.

Example 1. Evaluate $\cos(\sin^{-1} 3/5)$.

Solution. Let $\sin^{-1}(3/5) = y$, so that $\sin y = 3/5$ and $0 < y < \pi/2$.

Construct a right-angled triangle OAB in which AB = 3 units, OB = 5 units.

Then sin $\angle$ AOB = 3/5, so that the measure of $\angle$ AOB in radians is 3/5.

In the right-angled triangle OAB,

$OA^2 = OB^2 - AB^2 = 5^2 - 3^2 = 16$,

so that OA = 4 units.

Then cos $(\sin^{-1} 3/5) = \cos y$

$= \cos \angle AOB = OA/OB = 4/5$.

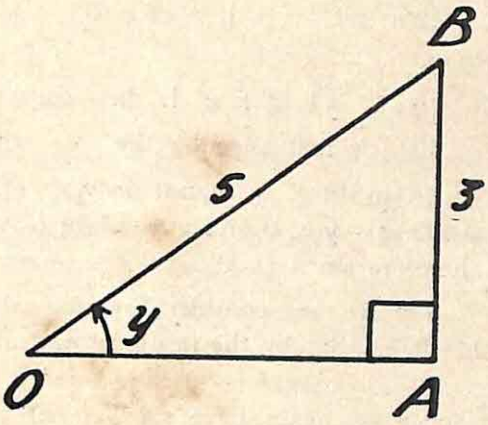

Fig. 8.11

Example 2. Show that $\cos^{-1} (12/13) = \tan^{-1} (5/12)$.

Solution. Let $\cos^{-1} (12/13) = y$. Then we have to show that tan y = 5/12 and $-\pi/2 \leqslant y \leqslant \pi/2$. Since 12/13 lies between 0 and 1, therefore y lies between 0 and $\pi/2$. The second condition is thus satisfied and we have only to prove that tan y = 5/12. Construct a right-angled triangle OAB in which OA = 12 units, OB = 13 units.

Now $AB^2 = OB^2 - OA^2$

$= (13)^2 - (12)^2$

$= 5^2$,

so that AB = 5 units.

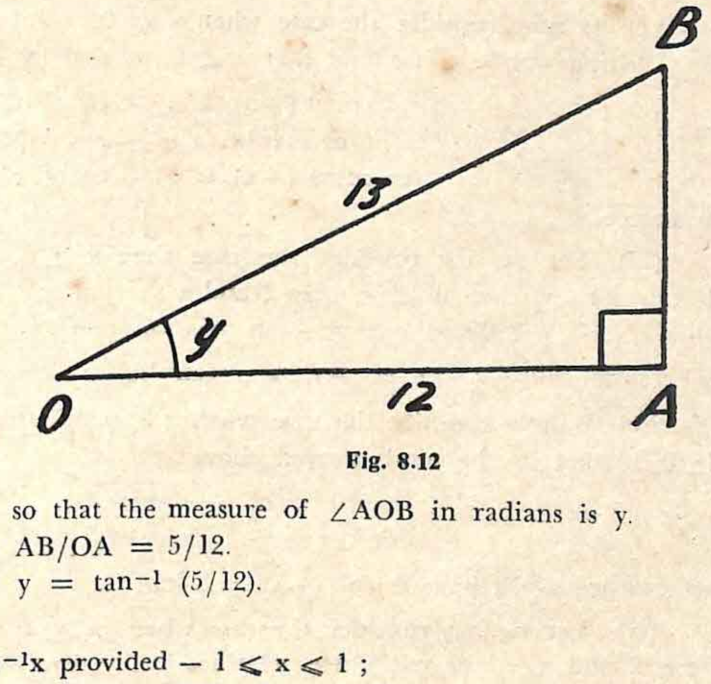

Fig. 8.12

Then cos $\angle$AOB = 12/13, so that the measure of $\angle$AOB in radians is y.

tan y = tan $\angle$AOB = AB/OA = 5/12.

Hence $\cos^{-1} (12/13) = y = \tan^{-1} (5/12)$.

Theorem 7.

(i) $\sin^{-1} (-x) = -\sin^{-1}x$ provided $-1 \leqslant x \leqslant 1$;

(ii) $\cos^{-1} (-x) = \pi - \cos^{-1}x$ provided $-1 \leqslant x \leqslant 1$;

(iii) $\tan^{-1}(-x) = -\tan^{-1}x$.

(iv) $\cot^{-1}(-x) = \pi - \cot^{-1}x$.

Proof.

(i) If $-1 \leqslant x \leqslant 1$, then $\sin^{-1}(-x)$ and $-\sin^{-1}x$ are both meaningful.

Let us first consider the case when $x \geq 0$.

If $\sin^{-1}x = y$, then $0 \leqslant y \leqslant \pi/2$ and $x = \sin y$. These relations imply that $-\pi/2 \leqslant -y \leqslant 0$ and $-x = \sin(-y)$.
Therefore $\sin^{-1}(-x) = -y = -\sin^{-1}x$.

Let us now consider the case when $x \leqslant 0$. Writing $-x = u$, we find that $u \geq 0$, so that by the result proved above

$$\sin^{-1}(-u) = -\sin^{-1}u, \text{ i.e., } \sin^{-1}x = -\sin^{-1}(-x),$$

whence we again have $\sin^{-1}(-x) = -\sin^{-1}x$.

(ii) If $-1 \leqslant x \leqslant 1$, then $\cos^{-1}(-x)$ and $-\cos^{-1}x$ are both meaningful.

Let us first consider the case when $x \geq 0$.

If $\cos^{-1}x = y$, then $0 \leqslant y \leqslant \pi/2$ and $x = \cos y$. These relations imply that $\pi/2 \leqslant \pi - y \leqslant \pi$ and $-x = -\cos y = \cos(\pi - y)$.
Therefore $\cos^{-1}(-x) = \pi - y = \pi - \cos^{-1}x$.

Let us now consider the case when $x \leqslant 0$.

Writing $-x = u$ we find that $u \leqslant 0$, so that by the result proved above

$$\cos^{-1}(-u) = \pi - \cos^{-1}u,$$
$$\text{or } \cos^{-1}x = \pi - \cos^{-1}(-x),$$
$$\text{or } \cos^{-1}(-x) = \pi - \cos^{-1}x,$$

as before.

(iii) Let us first consider the case when $x \geq 0$.
If $\tan^{-1}x = y$, then $0 \leqslant y < \pi/2$ and $x = \tan y$. These relations imply that $-\pi/2 < -y \leqslant 0$ and $-x = -\tan y = \tan(-y)$.
Therefore $\tan^{-1}(-x) = -y = -\tan^{-1}x$.

Let us now consider the case when $x \leqslant 0$. Writing $-x = u$ we find that $u \geq 0$, so that by the result proved above

$$\tan^{-1}(-u) = -\tan^{-1}u,$$
$$\text{or } \tan^{-1}x = -\tan^{-1}(-x),$$

whence we again have $\tan^{-1}(-x) = -\tan^{-1}x$.

(iv) Let us first consider the case when $x \geq 0$. If $\cot^{-1}x = y$, then $0 \leqslant y < \pi/2$ and $x = \cot y$. These relations imply that $-x = \cot(\pi - y)$.
Therefore $\cot^{-1}(-x) = \pi - y = \pi - \cot^{-1}x$.

Let us now consider the case when $x \leqslant 0$. Writting $-x = u$ we find that $u \geq 0$, so that by the result proved above

$$\begin{aligned} \cot^{-1}(-u) &= \pi - \cot^{-1} u, \\ \text{or } \cot^{-1} x &= \pi - \cot^{-1}(-x), \\ \text{or } \cot^{-1}(-x) &= \pi - \cot^{-1} x. \end{aligned}$$

Remark. The above theorem enables us to express $\sin^{-1} x$ in terms of $\sin^{-1}(|x|)$, $\cos^{-1} x$ in terms of $\cos^{-1}(|x|)$, $\tan^{-1} x$ in terms of $\tan^{-1}(|x|)$, etc. Also, for any given real number x, the values of expressions of the form $\cos(\tan^{-1} x)$, etc., can be computed by the use of right-angled triangles. Thus we can now compute $\cos(\tan^{-1} x)$, etc., whenever such an expression has a meaning.

Example 3. Evaluate $\cos(\sin^{-1}(-3/5))$.

Solution. Since $\sin^{-1}(-x) = -\sin^{-1} x$, therefore

$$\sin^{-1}(-3/5) = -\sin^{-1}(3/5).$$

$$\begin{aligned} \text{Now } \cos(\sin^{-1}(-3/5)) &= \cos(-\sin^{-1} 3/5), \\ &= \cos(\sin^{-1}(3/5)), \text{ since } \cos(-x) = \cos x, \\ &= 4/5, \text{ as in example 1.} \end{aligned}$$

Example 4. Show that $\cos^{-1}(-12/13) = \pi + \tan^{-1}(-5/12)$.

Solution. Since $\cos^{-1}(-x) = \pi - \cos^{-1} x$, therefore

$$\begin{aligned} \cos^{-1}(-12/13) &= \pi - \cos^{-1}(12/13), \\ &= \pi - \tan^{-1}(5/12), \text{ as in example 2,} \\ &= \pi + (-\tan^{-1}(5/12), \\ &= \pi + \tan^{-1}(-5/12), \text{ since } \tan^{-1}(-x) = -\tan^{-1} x. \end{aligned}$$

EXERCISES

Fill up the blanks in each of the following :

1. (i) $\sin^{-1}(-1)$ = ,
(ii) $\sin^{-1}(-\sqrt{3}/2)$ = ,
(iii) $\sin^{-1}(-1/\sqrt{2})$ = ,
(iv) $\sin^{-1}(-1/2)$ = ,
(v) $\sin^{-1} 0$ = ,
(vi) $\sin^{-1}(1/2)$ = ,
(vii) $\sin^{-1}(1\sqrt{2})$ = ,
(viii) $\sin^{-1}(\sqrt{3}/2)$ = ,
(ix) $\sin^{-1} 1$ = .

3. (i) $\tan^{-1}(-\sqrt{3})$ = ,
(ii) $\tan^{-1}(-1)$ = ,

2. (i) $\cos^{-1}(-1)$ = ,
(ii) $\cos^{-1}(-\sqrt{3}/2)$ = ,
(iii) $\cos^{-1}(-1/\sqrt{2})$ = ,
(iv) $\cos^{-1}(-1/2)$ = ,
(v) $\cos^{-1} 0$ = ,
(vi) $\cos^{-1}(1/2)$ = ,
(vii) $\cos^{-1} 1/\sqrt{2})$ = ,
(viii) $\cos^{-1}(\sqrt{3}/2)$ = ,
(ix) $\sin^{-1} 1$ = .

4. (i) $\cot^{-1}(-\sqrt{3})$ = ,
(ii) $\cot^{-1}(-1)$ = ,

(iii) $\tan^{-1}(-1\sqrt{3})$ = ,
(iv) $\tan^{-1} 0$ = ,
(v) $\tan^{-1}(1/\sqrt{3})$ = ,
(vi) $\tan^{-1} 1$ = ,
(vii) $\tan^{-1}\sqrt{3}$ = ,

(iii) $\cot^{-1}(-1/\sqrt{3})$ = ,
(iv) $\cot^{-1} 0$ = ,
(v) $\cot^{-1}(1/\sqrt{3})$ = ,
(vi) $\cot^{-1} 1$ = ,
(vii) $\cot^{-1}\sqrt{3}$ = ,

5. (i) $\sec^{-1}(-1)$ = ,
(ii) $\sec^{-1}(-2/\sqrt{3})$ = ,
(iii) $\sec^{-1}(-\sqrt{2})$ = ,
(iv) $\csc^{-1}(-2)$ = ,
(v) $\sec^{-1} 2$ = ,
(vi) $\sec^{-1}\sqrt{2}$ = ,
(vii) $\sec^{-1}(2/\sqrt{3})$ = ,
(viii) $\sec^{-1} 1$ = .

6. (i) $\csc^{-1}(-1)$ = ,
(ii) $\csc^{-1}(-2/\sqrt{3})$ = ,
(iii) $\csc^{-1}(-\sqrt{2})$ = ,
(iii) $\sec^{-1}(-\sqrt{2})$ = ,
(v) $\csc^{-1} 2$ = ,
(vi) $\csc^{-1}\sqrt{2}$ = ,
(vii) $\csc^{-1}(2/\sqrt{3})$ = ,
(viii) $\csc^{-1} 1$ = ,

Evaluate :

7. $\cos(\sin^{-1}(5/13))$.
8. $\sin(\tan^{-1}(24/7))$.
9. $\tan(\sin^{-1}(5/6))$.
10. $\cos(\cos^{-1}(4/9))$.
11. $\cos(\sin^{-1}(-4/5))$.
12. $\sin(\tan^{-1}(-5/7))$.
13. $\tan(\cos^{-1}(-5/11))$.
14. $\cot(\sin^{-1}(-24/25))$.

15. Show that $\sin^{-1}(4/5) = \tan^{-1}(4/3) = \cos^{-1}(3/5)$.

16. Show than $\sin^{-1}(-4/5) = \tan^{-1}(-4/3) = \cos^{-1}(-3/5) - \pi$.

17. Show that $\sin^{-1}x = \tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$, if $|x| \neq 1$.

18. Show that $\cos^{-1}x = \cot^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$, if $|x| \neq 1$.

19. Show that $\tan^{-1}x = \begin{cases} \cot^{-1}(1/x), & \text{if } x>0, \\ -\pi + \cot^{-1}(1/x), & \text{if } x<0. \end{cases}$

Fill in the blanks in problems 20—23. Assume $x>0$.

20. $\cos^{-1}\left\{\dfrac{x}{\sqrt{x^2+16}}\right\} = \sin^{-1}$........... $= \tan^{-1}$...........

21. $\sin^{-1}\left\{\dfrac{x}{\sqrt{x^2+9}}\right\} = \cos^{-1}$........... $= \tan^{-1}$...........

22. $\tan^{-1}\left\{\dfrac{x}{\sqrt{4-x^2}}\right\} = \sin^{-1}$........... $= \cos^{-1}$..........., $x>2$.

23. $\cot^{-1}\left\{\dfrac{\sqrt{x^2-9}}{3}\right\} = \sin^{-1}$........... $= \cos^{-1}$..........., $x>3$.

Draw the graph of the functions defined by the following equations :

24. $y = \frac{1}{2} \cos^{-1} x$.
25. $y = \frac{1}{2} \sin^{-1} x$.
26. $y = \frac{1}{2} \tan^{-1} x$.
27. $y = 2 \sin^{-1} x$.

Find from the tables the values of each of the following :

28. $\sin^{-1} (\cdot 6825)$.
29. $\cos^{-1} (\cdot 5960)$.
30. $\tan^{-1} (-1\cdot 298)$.
31. $\cot^{-1} (-\cdot 7155)$.

Example 5. Evaluate the expression

$$\sin (\cos^{-1} (3/5) + \tan^{-1} (-2)).$$

Solution. Let $\cos^{-1} (3/5) = x$, $\tan^{-1} (-2) = y$.

Then $\cos x = 3/5$, $0 \leqslant x \leqslant \pi/2$ and $\tan y = -2$, $-\pi/2 \leqslant y \leqslant 0$. Since $\sin^2 x = 1 - \cos^2 x = 1 - (3/5)^2 = (4/5)^2$, therefore $\sin x = \pm 4/5$. Since $0 \leqslant x \leqslant \pi/2$, sin x must be non-negative. Consequently $\sin x = 4/5$. $\cos^2 y = \dfrac{1}{1 + \tan^2 y} = \dfrac{1}{1 + (-2)^2} = 1/5$.

Since $-\pi/2 \leqslant y \leqslant 0$, therefore cos y must be non-negative.
Consequently $\cos y = 1/\sqrt{5}$.
Also, $\sin y = \tan y . \cos y = (-2). 1/\sqrt{5} = -2/\sqrt{5}$.

$$\begin{aligned} \text{Now } \sin \{\cos^{-1} (3/5) + \tan^{-1} (-2)\} &= \sin (x + y), \\ &= \sin x \cos y + \cos x \sin y, \\ &= 4/5. 1/\sqrt{5} + 3/5. (-2/\sqrt{5}), \\ &= -2/5\sqrt{5}. \end{aligned}$$

Example 6. Evaluate the expression

$$\tan (\sin^{-1} (-3/5) + \cot^{-1} 3).$$

Solution. Let $\sin^{-1} (-3/5) = x$, $\cot^{-1} 3 = y$

Then $\sin x = -3/5$, $-\pi/2 \leqslant x \leqslant 0$, and $\cot y = 3$, $0 < y \leqslant \pi/2$.

Since $-\pi/2 \leqslant x \leqslant 0$, therefore $\tan x \leqslant 0$. Also,

$$\tan^2 x = \frac{\sin^2 x}{1 - \sin^2 x} = \frac{(-3/5)^2}{1 - (-3/5)^2} = 9/16,$$

so that $\tan x = -3/4$.

Since $\cot y = 3$, therefore $\tan y = 1/3$.

$$\begin{aligned} \text{Now } \tan \{\sin^{-1} (-3/5) + \cot^{-} 3\} &= \tan (x + y), \\ &= \frac{\tan x + \tan y}{1 - \tan x \tan y}, \\ &= \frac{-3/4 + 1/3}{1 - (-3/4). 1/3} = -1/3. \end{aligned}$$

Example 7. Verify that

$$\tan^{-1}(1/2) + \tan^{-1}(1/3) = \pi/4.$$

Solution. Let $\tan^{-1}(1/2) = x$, $\tan^{-1}(1/3) = y$.

Since $0 < 1/2 < 1$, therefore $0 < \tan^{-1}(1/2) < \pi/4$, i.e., $0 < x < \pi/4$. Similarly $0 < y < \pi/4$. Therefore $0 < x + y < \pi/2$ ......(i)

Also, $\tan x = 1/2$, $\tan y = 1/3$.

$$\text{Now } \tan\{\tan^{-1}(1/2) + \tan^{-1}(1/3)\} = \tan(x+y),$$
$$= \frac{\tan x + \tan y}{1 - \tan x \tan y},$$
$$= \frac{1/2 + 1/3}{1 - 1/2.\,1/3},$$
$$= 1. \qquad \text{...........(ii)}$$

Since $\tan^{-1} 1/2 + \tan^{-1} 1/3$ lies between 0 and $\pi/2$, therefore we have from (ii),

$$\tan^{-1}(1/2) + \tan^{-1}(1/3) = \tan^{-1} 1 = \pi/4.$$

8.4. Sum Theorems

Theorem 8. If $-1 \leqslant x \leqslant 1$, then

(I) $\cos^{-1} x + \sin^{-1} x = \pi/2$

(II) $\tan^{-1} x + \cot^{-1} x = \pi/2$.

Proof. (I) Since $-1 \leqslant x \leqslant 1$, therefore each term in the left hand side is meaningful Let $x \geqslant 0$ and let $\cos^{-1} x = y$. Then $0 \leqslant y \leqslant \pi/2$ and $\cos y = x$.

Therefore $\sin(\pi/2 - y) = x$ and $0 \leqslant \pi/2 - y \leqslant \pi/2$.

Therefore $\sin^{-1} x = \pi/2 - y = \pi/2 - \cos^{-1} x$,

so that $\cos^{-1} x + \sin^{-1} x = \pi/2$. ....(i)

If $x < 0$, then $-x > 0$, and (i) gives

$$\cos^{-1}(-x) + \sin^{-1}(-x) = \pi/2.$$

But $\cos^{-1}(-x) = \pi - \cos^{-1} x$, and $\sin^{-1}(-x) = -\sin^{-1} x$.

Therefore $(\pi - \cos^{-1} x) + (-\sin^{-1} x) = \pi/2$,

so that $\cos^{-1} x + \sin^{-1} x = \pi/2$. ....(ii)

From (i) and (ii) we find that

If $-1 \leqslant x \leqslant 1$, then
$\cos^{-1} x + \sin^{-1} x = \pi/2$.

(II) Let us first consider the case when $x \geqslant 0$.

If $\tan^{-1} x = y$, then $x = \tan y$ and $0 \leqslant y < \pi/2$, i.e., $x = \tan y = \cot(\pi/2 - y)$, $0 < \pi/2 - y \leqslant \pi/2$.

Therefore $\cot^{-1} x = \pi/2 - y = \pi/2 - \tan^{-1} x$,
so that $\cot^{-1} x + \tan^{-1} x = \pi/2$, provided $x \geqslant 0$. ………(i)

Let us now consider the case when $x \leqslant 0$. Writing $x = -u$, we find that $u \geqslant 0$, so that by the result proved above we have

$$\tan^{-1}u + \cot^{-1}u = \pi/2,$$

$$\text{or } \tan^{-1}(-x) + \cot^{-1}(-x) = \pi/2.$$

But $\tan^{-1}(-x) = -\tan^{-1}x$, $\cot^{-1}(-x) = \pi - \cot^{-1}x$.
Therefore $(-\tan^{-1}x) + (\pi - \cot^{-1}x) = \pi/2$,
so that $\tan^{-1}x + \cot^{-1}x = \pi/2$. ………(ii)
From (i) and (ii) we find that for all $x \in \mathbf{R}$,

$$\tan^{-1}x + \cot^{-1}x = \pi/2.$$

Theorem 9.

(α) If $0 \leqslant xy < 1$, then

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}.$$

(β) If $xy = 1$, then

$$\tan^{-1}x + \tan^{-1}y = \begin{cases} \pi/2, \text{ if } x > 0, y > 0; \\ -\pi/2, \text{ if } x < 0, y < 0. \end{cases}$$

(γ) If $xy > 1$, then

$$\tan^{-1}x + \tan^{-1}y = \begin{cases} \pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right), \text{ if } x > 0, y > 0; \\ -\pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right), \text{ if } x < 0, y < 0. \end{cases}$$

(δ) If $x \geqslant 0, y \geqslant 0$, then

$$\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right).$$

Proof.

(α) Let $u = \tan^{-1} x$ and $v = \tan^{-1} y$.

If $xy = 0$, the proof is trivial. Let us therefore consider the case $0 < xy < 1$. Two different cases arise according as x and y are both positive or both negative.

Let us first assume that $x > 0$ and $y > 0$.
Then $0 < u < \pi/2$, $0 < v < \pi/2$, $x = \tan u$, $y = \tan v$.
Since $xy < 1$, therefore $x < 1/y$ i.e., $\tan u < 1/\tan v$, i.e., $\tan u < \tan (\pi/2 - v)$. ………(i)

Since tan is strictly increasing in $]0, \pi/2[$ and u, $\pi/2 - v$ both lie between 0 and $\pi/2$, therefore we have from (i)

$$u < \pi/2 - v,$$

so that

$$0 < u + v < \pi/2. \qquad \text{.........(ii)}$$

Therefore tan $(u + v)$ is defined.

$$\text{Now } \tan (u + v) = \frac{\tan u + \tan v}{1 - \tan u \tan v},$$

$$= \frac{x + y}{1 - xy}. \qquad \text{........ (iii)}$$

From (ii) and (iii) we have

$$u + v = \tan^{-1} \frac{x + y}{1 - xy}.$$

Thus $\tan^{-1}x + \tan^{-1}y = \tan^{-1} \dfrac{x + y}{1 - xy}$, provided $x > 0$, $y > 0$ and $xy < 1$.

............(iv)

Let now $x < 0$, $y < 0$. Then $-x > 0$, $-y > 0$, $(-x)(-y) < 1$.
From (iv) we have

$$\tan^{-1} (-x) + \tan^{-1} (-y) = \tan^{-1} \frac{(-x) + (-y)}{1-(-x)(-y)},$$

$$\text{or } \tan^{-1} (-x) + \tan^{-1} (-y) = \tan^{-1} \left\{-\left(\frac{x + y}{1 - xy}\right)\right\}. \qquad \text{.........(v)}$$

Since $\tan^{-1} (-x) = -\tan^{-1}x$, therefore we may write (v) as

$$(-\tan^{-1}x) + (-\tan^{-1}y) = -\tan^{-1} \left(\frac{x + y}{1 - xy}\right),$$

$$\text{or } \tan^{-1}x + \tan^{-1}y = \tan^{-1} \frac{x + y}{1 - xy}.$$

(β) Let $xy = 1$, $x > 0$, $y > 0$ and $\tan^{-1} x = u$.

Then $0 < u < \pi/2$ and $\tan u = x$. Therefore $1/x = \cot u$. Since $0 < u < \pi/2$, therefore

$$\cot^{-1} 1/x = u = \tan^{-1}x.$$

$$\begin{aligned}\text{Thus } \tan^{-1} x + \tan^{-1} y &= \cot^{-1} 1/x + \tan^{-1} y,\\ &= \cot^{-1} y + \tan^{-1} y,\\ &= \pi/2, \text{ by theorem 8.}\end{aligned} \qquad \text{......... (i)}$$

If $xy = 1$, $x < 0$, $y < 0$, then $-x > 0, -y > 0$, $(-x)(-y) = 1$, so that we have from (i)

$$\tan^{-1}(-x) + \tan^{-1} (-y) = \pi/2. \qquad \text{.........(ii)}$$

Since $\tan^{-1} (-x) = -\tan^{-1} x$, therefore (ii) may be written as

$$(-\tan^{-1} x) + (-\tan^{-1} y) = \pi/2,$$

or $\quad \tan^{-1} x + \tan^{-1} y = -\pi/2.$

(Υ) Let us first consider the case when $x > 0, y > 0$.
Let $\tan^{-1} x = u$ and $\tan^{-1} y = v$.
Then $x = \tan u, y = \tan v, 0 < u < \pi/2, 0 < v < \pi/2$.
Since $xy > 1$ and $y > 0$, therefore $x > 1/y$ i.e., $\tan u > \cot v$, i.e., $\tan u > \tan (\pi/2 - v)$. Since $0 < u < \pi/2, 0 < \pi/2 - v < \pi/2$, and since 'tan' is strictly increasing in $[0, \pi/2]$, therefore $u > \pi/2 - v$ or $u + v > \pi/2$. Also from $0 < u < \pi/2, 0 < v < \pi/2$, we have $0 < u + v < \pi$. Thus

$$\pi/2 < u + v < \pi,$$

or $\quad -\pi/2 < u + v - \pi < 0.$ ....(i)

Now $\tan (u + v - \pi) = -\tan (\pi - (u + v)),$

$$= \tan (u+v),$$

$$= \frac{\tan u + \tan v}{1-\tan u \tan v},$$

$$= \frac{x + y}{1-x y}. \qquad ....(ii)$$

From (i) and (ii) it follows that

$$\tan^{-1} \frac{x+y}{1-xy} = u + v - \pi = \tan^{-1} x + \tan^{-1} y - \pi,$$

$$\text{or } \tan^{-1} x + \tan^{-1} y = \pi + \tan^{-1} \frac{x+y}{1-xy}. \qquad ....(iii)$$

Let us now consider the case $x < 0, y < 0$.
Since $x < 0, y < 0, xy > 1$, therefore $-x > 0, -y > 0, (-x)(-y) > 1$.
From (iii) we then have

$$\tan^{-1} (-x) + \tan^{-1} (-y) = \pi + \tan^{-1} \frac{(-x)+(-y)}{1-(-x)(-y)}. \qquad ....(iv)$$

Since $\tan^{-1} (-x) = -\tan^{-1} x$, we may rewrite (iv) as

$$-\tan^{-1} x + (-\tan^{-1} y) = \pi - \tan^{-1} \frac{x + y}{1 - xy},$$

i.e., $\tan^{-1} x + \tan^{-1} y = -\pi + \tan^{-1} \dfrac{x + y}{1 - xy}$.

(δ) Let $\tan^{-1} x = u$ and $\tan^{-1} y = v$. Since $x \geqslant 0$,
$y \geqslant 0$, therefore $0 \leqslant u < \pi/2$ and $0 \leqslant v < \pi/2$ so that
$-\pi/2 < u - v < \pi/2$. .... (i)

Also $x = \tan u, y = \tan v$, so that

$$\tan (u-v) = \frac{x - y}{1 + xy}. \qquad ....(ii)$$

From (i) and (ii) it follows that

$$u - v = \tan^{-1} \frac{x - y}{1 + xy},$$

$$\tan^{-1}x - \tan^{-1}y = \tan^{-1} \frac{x - y}{1 + xy}.$$

Hence

$$0 \leqslant xy < 1 \Rightarrow \tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}; \qquad \ldots\ldots\ldots(\alpha)$$

$$xy=1 \Rightarrow \tan^{-1}x + \tan^{-1} y = \begin{cases} \pi/2, \text{ if } x > 0,\ y > 0, \\ -\pi/2, \text{ if } x < 0,\ y < 0, \end{cases} \qquad \ldots\ldots\ldots(\beta)$$

$$xy > 1 \Rightarrow \tan^{-1}x + \tan^{-1} y = \begin{cases} \pi + \tan^{-1} \dfrac{x+y}{1-xy} \text{ if } x > 0,\ y > 0, \\ -\pi + \tan^{-1} \dfrac{x+y}{1-xy} \text{ if } x < 0,\ y < 0 \end{cases} \qquad \ldots\ldots\ldots(\Upsilon)$$

$$x \geqslant 0,\ y \geqslant 0 \Rightarrow \tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x-y}{1+xy}. \qquad \ldots\ldots\ldots(\delta)$$

Corollary. If $-1 < x < 1$, then

$$2 \tan^{-1}x = \tan^{-1} \frac{2x}{1 - x^2}.$$

Proof. Put $y = x$ in (α).

Example 8. Prove that

(i) $\tan^{-1} 1/2 + \tan^{-1} 1/3 = \pi/4$;
(ii) $\tan^{-1} 1/2 + \tan^{-1} 2 = \pi/2$;
(iii) $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$.

Solution. (i) Let $x = 1/2$, $y = 1/3$. Then $xy < 1$, so that by (α),

$$\tan^{-1} 1/2 + \tan^{-1} 1/3 = \tan^{-1}x + \tan^{-1}y,$$

$$= \tan^{-1} \frac{x + y}{1 - xy},$$

$$= \tan^{-1} \left(\frac{1/2+1/3}{1-1/2.1/3} \right) = \tan^{-1} (1) = \pi/4.$$

(ii) Let $x = 1/2$, $y = 2$. Then $x > 0$, $y > 0$, $xy = 1$, so that by (β),

$$\tan^{-1} 1/2 + \tan^{-1} 2 = \pi/2.$$

(iii) Let $x = 2$, $y = 3$. Then $x > 0$, $y > 0$, $xy > 1$, so that by (Υ),

$$\tan^{-1} 2 + \tan^{-1} 3 = \pi + \tan^{-1} \left(\frac{2+3}{1-2.3} \right),$$

$$= \pi + \tan^{-1} (-1),$$

$$= \pi - \tan^{-1} 1,$$

whence $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$

Example 9. Prove that

$$4 \tan^{-1} (1/5) - \tan^{-1} (1/239) = \pi/4.$$

Solution. By corollary to theorem 9,

$$2 \tan^{-1} 1/5 = \tan^{-1} \frac{2.1/5}{1-(1/5)^2} = \tan^{-1} (5/12).$$

Again by the same corollary,

$$2 \tan^{-1} (5/12) = \tan^{-1}\left(\frac{2.5/12}{1-(5/12)^2}\right) = \tan^{-1} (120/119).$$

$$\begin{aligned} \text{Now } 4 \tan^{-1} 1/5 - \tan^{-1} 1/239 &= 2\,(2 \tan^{-1} 1/5) - \tan^{-1} 1/239, \\ &= 2 \tan^{-1} 5/12 - \tan^{-1} 1/239, \\ &= \tan^{-1} 120/119 - \tan^{-1} 1/239. \qquad \ldots\text{(i)} \end{aligned}$$

Since $120/119 > 0$, $1/239 > 0$, therefore by theorem 9 (δ), the right-hand members of (i) becomes

$$\tan^{-1} \frac{120/119 - 1/239}{1 + 120/119.1/239} = \tan^{-1} 1 = \pi/4.$$

Hence $4 \tan^{-1} 1/5 - \tan^{-1} 1/239 = \pi/4$.

8.5. Inverse Trigonometric Equations

The following examples will illustrate the method of solving equations involving inverse trigonometric functions.

Example 10. Determine the solution set of the equation

$$2 \sin^{-1} x = \cos^{-1} (1 - 2x^2).$$

Solution. $\sin^{-1} x$ is meaningful provided $-1 \leqslant x \leqslant 1$.
$\cos^{-1} (1-2x^2)$ is meaningful provided $-1 \leqslant 1 - 2x^2 \leqslant 1$,
i.e., provided $0 \leqslant x^2 \leqslant 1$, i.e., provided $|x| \leqslant 1$. Thus both sides of the given equation are meaningful when x belongs to the set

$$A = \{a : -1 \leqslant a \leqslant 1\}.$$

Also, if $x \in A$, then $\cos^{-1} (1 - 2x^3)$ lies between 0 and π. Therefore if the equality has to hold, then we must have

$$0 \leqslant 2 \sin^{-1} x \leqslant \pi,$$

i.e., $\qquad 0 \leqslant \sin^{-1} x \leqslant \pi/2,$

i.e., $\qquad 0 \leqslant x \leqslant 1,$

i.e., x belongs to the set

$$B = \{a : 0 \leqslant a \leqslant 1\}.$$

We shall now show that if
$x \in A \cap B = \{a : 0 \leqslant a \leqslant 1\}$, then the given equation is satisfied.

Let $\sin^{-1} x = y$ and let $x \in A \cap B$. Then $x = \sin y$, and $0 \leqslant y \leqslant \pi/2$. (y must always lie between $-\pi/2$ and $\pi/2$, and since x is non-negative, therefore y cannot be negative).

Therefore $\cos 2y = 1 - 2 \sin^2 y = 1 - 2x^2$, and $0 \leqslant 2y < \pi$.

Thus $2y = \cos^{-1} (1 - 2x^2)$,

or $2 \sin^{-1} x = \cos^{-1} (1-2x^2)$.

Hence the solution-set of the given equation is $\{ a : 0 \leqslant a < 1 \}$.

Example 11. Determine the solution set of the equation

$$\sin^{-1} (2\sqrt{x}) = \cos^{-1} x.$$

Solution. Since $\sqrt{x}$ occurs in the left-hand side of the given equation, therefore $x \geqslant 0$. Also, if $x \geqslant 0$, then $\cos^{-1} x$ lies between 0 and $\pi/2$.

Writing $\sin^{-1} (2\sqrt{x}) = \cos^{-1} x = y$, we have

$$2\sqrt{x} = \sin y, \ x = \cos y.$$

Since $\cos^2 y + \sin^2 y = 1$, therefore we must have

$$x^2 + 4x = 1. \qquad \text{.........(i)}$$

Solving the equation (i), we have

$$x = -2 \pm \sqrt{5}.$$

Since $x \geqslant 0$, therefore we must have $x = \sqrt{5} - 2$. We shall now verify that $x = \sqrt{5} - 2$ is actually a solution of the given equation.

Now $x = \sqrt{5} - 2 \Rightarrow (x + 2)^2 = 5,$

$$\Rightarrow 4x = 1 - x^2,$$

$$\Rightarrow 2\sqrt{x} = \sqrt{1 - x^2}, \text{ since x is positive,}$$

$$\Rightarrow \sin^{-1} (2\sqrt{x}) = \sin^{-1} \{ \sqrt{(1 - x^2)} \}. \qquad \text{.........(ii)}$$

Also $\sin^{-1} (\sqrt{1-x^2}) = \cos^{-1} x$, wherever $0 < x < 1$. ....(iii)

From (ii) and (iii) we have

$$\sin^{-1} (2\sqrt{x}) = \cos^{-1} x, \text{ when } x = \sqrt{5}-2.$$

Therefore $x = \sqrt{5} - 2$ is the solution.

Example 12. Solve :

$$\cos^{-1} x + \cos^{-1} y = \pi/2;$$
$$\tan^{-1} x - \tan^{-1} y = 0.$$

Solution. From the second of the given equations we have $x = y$. ....(i)

Substituting $x = y$ in the first of the given equations, we have

$$2 \cos^{-1} x = \pi/2,$$

or $\cos^{-1} x = \pi/4,$

or $x = \cos \pi/4 = 1/\sqrt{2}$ ....(ii)

From (i) and (ii) we have $x = 1/\sqrt{2}$, $y = 1/\sqrt{2}$. We can easily see that these values actually satisfy the given equations. Hence the solution set of the given equations is $\{ (1/\sqrt{2}, 1/\sqrt{2}) \}$,

EXERCISES

1. Evaluate each of the following expressions :
   - (i) $\sin(\cos^{-1} 4/5 + \tan^{-1} 12/5)$.
   - (ii) $\cos(\sin^{-1} 8/15 + \cot^{-1} 4/3)$.
   - (iii) $\sin(\cos^{-1}(-1/2) + \tan^{-1}(1\sqrt{3}))$.
   - (iv) $\cos\{\sin^{-1}(-24/25) + \tan^{-1}(5/12)\}$.
   - (v) $\tan(\sin^{-1} 8/17 + \cot^{-1} 8/15)$.
   - (vi) $\tan\{\cos^{-1}(-1/2) + \sin^{-1}(-\sqrt{3}/2)\}$.
2. Verify each of the following statements :
   - (i) $\tan^{-1}(2/3) + \tan^{-1}(1/5) = \pi/4$.
   - (ii) $\tan^{-1}(4/3) + \tan^{-1} 7 = 3\pi/4$.
   - (iii) $\cos^{-1}(11/14) + \sin^{-1}(3\sqrt{3}/14) = \pi/3$.
   - (iv) $\sin^{-1}(1/\sqrt{82}) + \sin^{-1}(4/\sqrt{41}) = \pi/4$.
   - (v) $\sin^{-1}(59/91) + \sin^{-1}(1/7) = \cos^{-1} 11/13$.
   - (vi) $\sin^{-1}(5/13) + \cos^{-1}(15/17) = \tan^{-1}(171/140)$.
   - (vii) $\sin^{-1}(5/13) + \sin^{-1}(33/65) = \sin^{-1}(4/5)$.
   - (viii) $\sin^{-1}(3/5) + \tan^{-1}(63/16) = \pi - \cos^{-1}(5/13)$.
   - (ix) $\tan^{-1}(1/3) + \tan^{-1}(1/5) + \tan^{-1}(1/7) + \tan^{-1}(1/8) = \pi/4$.
3. Prove.
   - (i) $\sin(\cos^{-1} x) = \sqrt{(1 - x^2)}$, if $|x| \leqslant 1$.
   - (ii) $\cos(\tan^{-1} x) = \dfrac{1}{\sqrt{(1 + x^2)}}$.
   - (iii) $\sin(\tan^{-1} x) = \dfrac{x}{\sqrt{(1 + x^2)}}$.
   - (iv) $\sin^{-1} x + \sin^{-1}\sqrt{(1 - x^2)} = \pi/2$ proved $x \geqslant 0$.
   - (v) $\sin((1/2)\cos^{-1} x) = \sqrt{\left(\dfrac{1 - x}{2}\right)}$.
4. Determine the solution set of each of the following equations :
   - (i) $\tan(\cos^{-1} x) = \dfrac{\sqrt{(1 - x^2)}}{x}$.
   - (ii) $\cos(\sin^{-1} x) = \sqrt{(1 - x^2)}$.
   - (iii) $\tan(\sin^{-1} x) = \dfrac{x}{\sqrt{(1 - x^2)}}$.
   - (iv) $2\tan^{-1} x = \cos^{-1}\dfrac{1 - x^2}{1 + x^2}$.
   - (v) $2\tan^{-1} x = \sin^{-1}\dfrac{2x}{1 + x^2}$.
   - (vi) $2\tan^{-1} x = \tan^{-1}\dfrac{2x}{1 - x^2}$.

(vii) $2 \cos^{-1} x = \cos^{-1} (2x^2-1)$.

(viii) $3 \sin^{-1} x = \sin^{-1} (3x-4x^3)$.

(ix) $\tan^{-1} 2x + \tan^{-1} 3x = \pi/4$.

(x) $\sin^{-1} x + \sin^{-1} (2x) = \pi/3$.

ASSORTED EXERCISES ON CHAPTER 8

1. Evaluate each of the following :
   - (i) $\cos (\tan^{-1} 3/4)$.
   - (ii) $\tan \{\sin^{-1} (-1/\sqrt{5})\}$.
   - (iii) $\sin \{\tan^{-1} (-4/3) + \sec^{-1} (13/5)\}$.
   - (iv) $\tan \{\sin^{-1} (-5/13) + \cos^{-1} (8/17)\}$.
2. Express in terms of x :
   - (i) $\cos (\frac{1}{2} \sin^{-1} 2x)$. (ii) $\sin^2 (\tan^{-1} x)$.
   - (iii) $\tan^2 \{\sec^{-1} \sqrt{(1 + x^2)}\}$. (iv) $\cos \{ (1/2) \cos^{-1} x) \}$.
3. Prove each of the following identities:
   - (i) $\tan^{-1} 1/7 + 2 \tan^{-1} 1/3 = \pi/4$.
   - (ii) $\sin^{-1} 3/5 + \sin^{-1} 5/13 = \tan^{-1} 56/33$.
   - (iii) $\tan^{-1} 1/8 + \tan^{-1} 1/5 = \tan^{-1} 1/3$.
   - (iv) $\sin^{-1} 4/5 + \sin^{-1} 5/13 + \sin^{-1} 16/65 = \pi/2$.
4. Solve each of the following equations:
   - (i) $\cos^{-1} x + \sin^{-1} (1 - x) = 0$.
   - (ii) $\cos^{-1} x + \sin^{-1} (1 - x) = \pi/2$.
   - (iii) $\tan^{-1} x + \tan^{-1} 3x = \cot^{-1} 1/2$.
   - (iv) $2 \cos^{-1} x = \sin^{-1} (1 - 2x^2)$.
5. Sketch the graph of the following functions :
   - (i) $\{ (x, y) : y = \cot^{-1} x \}$. (ii) $\{ (x, y) : y = 2 \cos^{-1} x \}$.
   - (iii) $\{ (x, y) : y = \sec^{-1} x \}$. (iv) $\{ (x, y): y = \csc^{-1} x \}$.
6. Is the following statement true :
   $\tan^{-1} 2 + \tan^{-1} 3 = -\pi/4$?
7. Is it meaningful to talk of $\cos^{-1} 3$?
8. Does it make sense to talk of $\sin^{-1} (-4)$?
9. Discuss whether the equation

$$2 \tan^{-1} x = \sin^{-1} \frac{2x}{1 + x^2}$$

is an identity for all x in $[-1, 1]$.

Does it hold for any value of x such that $| x | > 1$.

10. Solve the inequality $\left| \cos^{-1} \left(\frac{1-x^2}{1+x^2}\right) \right| < \pi/3$.

Chapter 9

Trigonometric Equations

9.1. Introduction

In the preceding chapter we were primarily concerned with problem of the following type :

To determine x such that $-\pi/2 \leqslant x \leqslant \pi/2$ *and* $\sin x = k$, *k being a given real number* : or equivalently, to find that solution of $\sin x = k$ which lies in the interval $[-\pi/2, \pi/2]$.

In the present chapter we shall devote ourselves to the more general problem of finding *all* solutions of the equation $\sin x = k$ (and other similar equations of course !), k being a given real number. We shall, however, first take up some special cases.

While we shall be dealing with trigonometric functions of numbers, the results will apply equally well to trigonometric functions of angles.

9.2. To Solve the Equation $\sin x = 0$

In the interval $[-\pi/2, \pi/2]$, $\sin x$ increases strictly from -1 to $+1$ and assumes the value 0 only once, viz. at $x = 0$. In the interval $[\pi/2, 3\pi/2]$, $\sin x$ decreases strictly from $+1$ to -1 and assumes the value 0 only once at $x = \pi$.

By periodicity all the values of x, for which $\sin x = 0$, are given by $2p\pi$ and $2p\pi + \pi$, where p is any integer, so that

$$\{x : \sin x = 0\} = \{2p\pi : p \in \mathbf{Z}\} \cup \{(2p+1)\pi : p \in \mathbf{Z}\},$$
$$= \{n\pi : n \in \mathbf{Z}\}.$$

Hence

$$\boxed{\{x : \sin x = 0\} = \{n\pi : n \in \mathbf{Z}\}}$$

Remark. **Z** will stand throughout for the set of all integers.

9.3. To Solve the Equation cos x = 0

In the interval $[0, \pi]$, $\cos x$ decreases strictly from $+1$ to -1 and assumes the value 0 exactly once, viz. at $x = (1/2)\pi$. Similarly in the interval $[\pi, 2\pi[$ it assumes the value 0 only once at $x = (3/2)\pi$. By periodicity all the values of the variable x, for which $\cos x = 0$, are given by $2k\pi + \pi/2$, $2k\pi + 3\pi/2$, where k is any integer, so that

$$\{x : \cos x = 0\} = \{2k\pi + \pi/2 : k \in \mathbf{Z}\} \cup \{2k\pi + 3\pi/2 : k \in \mathbf{Z}\},$$
$$= \{(2n+1)(\pi/2) : n \in \mathbf{Z}\}.$$

Hence

$$\boxed{\{x : \cos x = 0\} = \{(2n+1)\pi/2 : n \in \mathbf{Z}\}}$$

EXERCISES

1. Solve the equation $\tan x = 0$.
2. Solve the equation $\cot x = 0$.
3. Imitate the method of Sec. 9.1 and 9.2 to solve the following equations:

   (i) $\sin x = 1/2$.

   (ii) $\sin x = \dfrac{-\sqrt{3}}{2}$

   (iii) $\sin x = -\dfrac{1}{\sqrt{2}}$

   (iv) $\sin x = -\frac{1}{2}$.

   (v) $\cos x = 1$.

   (iv) $\cos x = -\frac{1}{2}$.

   (vii) $\cos x = \dfrac{-\sqrt{3}}{2}$

   (viii) $\cos x = \dfrac{\sqrt{3}}{2}$

9.4. To Solve the Equation sin x = k, k being a fixed real number in [−1, 1].

Let $\alpha = \sin^{-1} k$. In the interval $[-\pi/2, \pi/2]$, $\sin x$ increases strictly from -1 to 1 and assumes the value k only once, viz. at α. In the interval $[\pi/2, 3\pi/2]$, $\sin x$ decreases strictly from 1 to -1 and assumes the value only once. In particular, it assumes the value k only once, namely at $x = \pi - \alpha$. In fact, $\sin(\pi - \alpha) = \sin \alpha = k$; also, since $-\pi/2 \leqslant \alpha \leqslant \pi/2$, therefore $\pi/2 \leqslant \pi - \alpha \leqslant 3\pi/2$.

By periodicity of sin x, all the values of x, for which $\sin x = k$, are given by $2p\pi + \alpha$, $2p\pi + (\pi - \alpha)$, where p is any integer, so that

$$\begin{aligned}\{x : \sin x = k\} &= \{2p\pi + \alpha : p \in \mathbf{Z}\} \cup \{2p\pi + (\pi - \alpha) : p \in \mathbf{Z}\},\\ &= \{n\pi + (-1)^n \alpha : n \in \mathbf{Z}\},\\ &= \{n\pi + (-1)^n \sin^{-1} k : n \in \mathbf{Z}\}.\end{aligned}$$

Hence $$\boxed{\{x : \sin x = k\} = \{n\pi + (-1)^n \sin^{-1} k : n \in \mathbf{Z}\}}$$

Alternatively, we may proceed as follows :

Let $\alpha = \sin^{-1} k$. Then the given equation can be written as

$$\sin x = \sin \alpha,$$

or $$\sin x - \sin \alpha = 0.$$

$$\begin{aligned}\text{Now } \sin x - \sin \alpha = 0 &\Rightarrow 2\cos\{(1/2)(x + \alpha)\}\sin\{(1/2)(x - \alpha)\} = 0,\\ &\Rightarrow \cos\{(1/2)(x + \alpha)\} = 0 \vee \sin\{(1/2)(x - \alpha)\} = 0,\\ &\Rightarrow (1/2)(x + \alpha) \in \{(2n+1)\pi/2 : n \in \mathbf{Z}\} \vee \tfrac{1}{2}(x - \alpha) \in \{n\pi : n \in \mathbf{Z}\},\\ &\Rightarrow x \in \{(2n+1)\pi - \alpha : n \in \mathbf{Z}\} \vee\\ &\quad\;\; x \in \{2n\pi + \alpha : n \in \mathbf{Z}\},\\ &\Rightarrow x \in \{m\pi + (-1)^m \alpha : m \in \mathbf{Z}\} \qquad \vee \ldots\ldots \text{(i)}\end{aligned}$$

$$\begin{aligned}\text{Also, } x \in \{m\pi + (-1)^m \alpha : m \in \mathbf{Z}\} &\Rightarrow \sin x = \sin(m\pi + (-1)^m \alpha), \text{ where } m \in \mathbf{Z},\\ &\Rightarrow \sin x = (-1)^m \sin\{(-1)^m \alpha\},\\ &\Rightarrow \sin x = \sin \alpha. \qquad \ldots\ldots \text{(ii)}\end{aligned}$$

From (i) and (ii) we find that

$$\sin x - \sin \alpha = 0 \Longleftrightarrow x \in \{m\pi + (-1)^m \alpha : m \in \mathbf{Z}\}.$$

Hence $\{x : \sin x = k\} = \{m\pi + (-1)^m \sin^{-1} k : m \in \mathbf{Z}\}$.

Remark. From the above working (and even otherwise) it can be easily seen that if α be *any* solution of the equation $\sin x = k$ (i.e. if α is not necessarily $\sin^{-1} k$, but any number such that $\sin \alpha = k$), even then the solution-set of the equation $\sin x = k$ is $\{m\pi + (-1)^m \alpha : m \in \mathbf{Z}\}$. This observation would be found extremely useful in problems later on.

9.5. To Solve the Equation cos x = k, k being a fixed real number in [− 1, 1].

Let $\alpha = \cos^{-1} k$. In the interval $[0, \pi]$, cos x decreases strictly from $+1$ to -1 and assumes each value between 1 and -1 exactly once. In particular, it assumes the value k only at α. In the interval $[\pi, 2\pi]$, cos x increases strictly from -1 to 1 and assumes each between -1 and 1 exactly once. In particular, it assumes the value k only once, namely at $x = 2\pi - \alpha$. In fact, $\cos(2\pi - \alpha) = \cos\alpha = k$; also, since $0 < \alpha < \pi$, therefore $\pi < 2\pi - \alpha < 2\pi$.

By periodicity of cos x, all the values of x for which $\cos x = k$ are given by $2p\pi + \alpha$, $2p\pi + (2\pi - \alpha)$, where p is any integer, so that

$$\begin{aligned}\{x : \cos x = k\} &= \{2p\pi + \alpha : p \in \mathbf{Z}\} \cup \{2p\pi + (2\pi - \alpha) : p \in \mathbf{Z}\},\\ &= \{2n\pi \pm \alpha : n \in \mathbf{Z}\},\\ &= \{2n\pi \pm \cos^{-1} k : n \in \mathbf{Z}\}.\end{aligned}$$

Hence $\boxed{\{x : \cos x = k\} = \{2n\pi \pm \cos^{-1} k : n \in \mathbf{Z}\}}$

Alternatively, we may proceed as follows :

Let $\alpha = \cos^{-1} k$. Then the given equation can be written as

$$\cos x = \cos\alpha,$$

$$\text{or } \cos x - \cos\alpha = 0.$$

$$\begin{aligned}\text{Now } \cos x - \cos\alpha = 0 &\Rightarrow -2\sin\{(1/2)(x+\alpha)\}\sin\{(1/2)(x-\alpha)\} = 0,\\ &\Rightarrow \sin\{(1/2)(x+\alpha)\} = 0 \vee \sin\{(1/2)(x-\alpha)\} = 0,\\ &\Rightarrow (1/2)(x+\alpha) \in \{n\pi : n \in \mathbf{Z}\} \vee (1/2)(x-\alpha) \in \{n\pi : n \in \mathbf{Z}\},\\ &\Rightarrow x \in \{2n\pi - \alpha : n \in \mathbf{Z}\} \vee x \in \{2n\pi + \alpha : n \in \mathbf{Z}\},\\ &\Rightarrow x \in \{2n\pi \pm \alpha : n \in \mathbf{Z}\}. \qquad \ldots\ldots\ldots(\text{i})\end{aligned}$$

$$\begin{aligned}\text{Also, } x \in \{2n\pi \pm \alpha : n \in \mathbf{Z}\} &\Rightarrow \cos x = \cos(2n\pi \pm \alpha), \text{ where } n \in \mathbf{Z},\\ &\Rightarrow \cos x = \cos\alpha. \qquad \ldots\ldots\ldots(\text{ii})\end{aligned}$$

From (i) and (ii) we find that

$$\cos x - \cos\alpha = 0 \Longleftrightarrow x \in \{2n\pi \pm \alpha : n \in \mathbf{Z}\}.$$

Hence $\{x : \cos x = k\} = \{2n\pi \pm \cos^{-1} k : n \in \mathbf{Z}\}$.

Remark. If α be *any* solution (*whatsoever*) of the equation $\cos x = k$, even then the solution-set of the equation can be easily seen to be $\{2n\pi \pm \alpha : n \in \mathbf{Z}\}$.

9.6. To Solve the Equation tan x = k, k being any fixed real number

Let $\alpha = \tan^{-1} k$. In the interval $[-\pi/2, \pi/2]$, tan x increases strictly and assumes each value exactly once.

In particular, it assumes the value k only at α.

By periodicity of tan x, all the values of x for which $\tan x = k$ are given by $n\pi + \alpha$ where n is any integer, so that

$$\begin{aligned}\{x : \tan x = k\} &= \{n\pi + \alpha : n \in \mathbf{Z}\}.\\ &= \{n\pi + \tan^{-1} k : n \in \mathbf{Z}\}.\end{aligned}$$

Henee $\boxed{\{x : \tan x = k\} = \{n\pi + \tan^{-1} k : n \in \mathbf{Z}\}}$

Alternatively, we may proceed as follows :

Let $\alpha = \tan^{-1} k$. Then the given equation can be written as

$$\tan x = \tan \alpha .$$

Now $\tan x = \tan \alpha \Rightarrow \dfrac{\sin x}{\cos x} - \dfrac{\sin \alpha}{\cos \alpha} = 0,$

$$\Rightarrow \frac{\sin (x - \alpha)}{\cos x \cos \alpha} = 0,$$

$$\Rightarrow \sin (x - \alpha) = 0, \quad \text{(Why ?)}$$

$$\Rightarrow (x - \alpha) \in \{ n\pi : n \in \mathbf{Z} \},$$

$$\Rightarrow x \in \{ n\pi + \alpha : n \in \mathbf{Z} \}. \qquad \text{.........(i)}$$

Also, $x \in \{ n\pi + \alpha : n \in \mathbf{Z} \} \Rightarrow \tan x = \tan (n\pi + \alpha),$

$$\Rightarrow \tan x = \tan \alpha .$$

From (i) and (ii) we find that

$$\tan x = \tan \alpha \Longleftrightarrow x \in \{ n\pi + \alpha : n \in \mathbf{Z} \}.$$

Hence $\{ x : \tan x = k \} = \{ n\pi + \tan^{-1} k : n \in \mathbf{Z} \}$.

Remark. It can be easily seen that if α be *any* solution whatsoever of the equation $\tan x = k$, then the solution-set of the equation can be written as $\{ n\pi + \alpha : n \in \mathbf{Z} \}$

Corollary. If $k \neq 0$, then the solution-set of the equation $\cot x = k$ is $\{ n\pi + \cot^{-1} k : n \in \mathbf{Z} \}$.

Proof. $\cot x = k \Longleftrightarrow \tan x = 1/k, \; (k \neq 0)$

$$\Longleftrightarrow x \in \{ n\pi + \tan^{-1} 1/k : n \in \mathbf{Z} \}$$

$$\Longleftrightarrow x \in \{ n\pi + \cot^{-1} k : n \in \mathbf{Z} \}, \text{ since } \tan^{-1} \frac{1}{k} = \cot^{-1} k.$$

Example 1. Find the solution-sets of the following equations :

(i) $\sin x = 1/2$. (ii) $\cos x = 1/3$.

(iii) $\tan x = -\sqrt{3}$. (iv) $\cot x = -4$.

(v) $\sec x = \sqrt{2}$. (iv) $\csc x = 7$.

Solution. (i) Since $\sin^{-1} (1/2) = \pi/6$, therefore the solution-set of the equation $\sin x = 1/2$ is

$$\{ n\pi + (-1)^n \sin^{-1} 1/2 : n \in \mathbf{Z} \},$$

i.e., $\{ n\pi + (-1)^n \pi/6 : n \in \mathbf{Z} \}$.

(ii) The solution-set of the equation $\cos x = 1/3$ is $\{ 2n \pm \cos^{-1} (1/3) : n \in \mathbf{Z} \}$.

(iii) Since $\tan^{-1} (\sqrt{3}) = -\pi/3$, therefore the solution-set of the equation $\tan x = -\sqrt{3}$ is

$\{ n\pi + \tan^{-1} (-\sqrt{3}) : n \in \mathbf{Z} \}$, i.e., $\{ n\pi + (-\pi/3) : n \in \mathbf{Z} \}$,

i.e., $\{ n\pi - \pi/3 : n \in \mathbf{Z} \}$.

(iv) $\cot x = -4 \iff \tan x = -\frac{1}{4} \iff x \in \{ n\pi + \tan^{-1}(-\frac{1}{4}) : n \in \mathbf{Z} \}$.

Hence the solution-set of the given equation is $\{ n\pi + \tan^{-1}(-1/4) : n \in \mathbf{Z} \}$.

(v) $\sec x = -\sqrt{2} \iff \cos x = -\dfrac{1}{\sqrt{2}} \iff x \in \{2n\pi \pm \cos^{-1}\left(\dfrac{1}{\sqrt{2}}\right) : n \in \mathbf{Z} \}$,

$$\iff x \in \{ 2n\pi \pm \frac{3\pi}{4} : n \in \mathbf{Z} \}.$$

Hence the solution-set of the given equation is

$$\{ 2n\pi \pm 3\pi/4 : n \in \mathbf{Z} \}.$$

(vi) $\csc x = 7 \iff \sin x = 1/7$,

$$\iff x \in \{ n\pi + (-1)^n \sin^{-1}\tfrac{1}{7} : n \in \mathbf{Z} \}.$$

Hence the solution-set of the given equation is

$$\{ n\pi + (-1)^n \sin^{-1}(\tfrac{1}{7}) : n \in \mathbf{Z} \}.$$

EXERCISES

1. Determine without using any formula, the solution-set of each of the following equations :

(i) $\sin x = \dfrac{\sqrt{3}}{2}$.

(ii) $\cos x = -1/2$.

(iii) $\tan x = 1$.

(iv) $\cot x = -\sqrt{3}$.

(v) $\sec x = 2$.

(vi) $\csc x = -2$.

(vii) $\cos x = 1/4$.

(viii) $\tan x = 5$.

2. Write down the solution-set of each of the following equations :

(i) $\sin x = -\dfrac{1}{\sqrt{2}}$

(ii) $\cos x = \dfrac{\sqrt{3}}{2}$.

(iii) $\tan 2x = 1/\sqrt{3}$.

(iv) $\cot 3x = -2$.

(v) $\sec x = \sqrt{2}$.

(vi) $\csc x = -6$.

3. Determine all angles whose cosine is $-\dfrac{\sqrt{3}}{2}$.

4. Determine all angles θ such that $\tan\theta = -1$.

5. Solve the following equations:

(i) $2\cos x - 1 = 0$.

(ii) $\sqrt{2}\sin x + 1 = 0$.

(iii) $\sqrt{3}\tan 2x + 1 = 0$.

(iv) $5\cot 4x + 2 = 0$.

6. Verify that if α be *any* solution of the equation $\sin x = k$, then the solution-set of the equation can be expressed as $\{ n\pi + (-1)^n \alpha : n \in \mathbf{Z} \}$.

7. Verify that if α be *any* solution of the equation $\cos x = k$, then the solution-set of the equation can be expresed as $\{ 2n\pi \pm \alpha : n \in \mathbf{Z} \}$.

8. Verify that if α be *any* solution of the equation $\tan x = k$, then the solution-set of the equation can be expressed as $\{ n\pi + \alpha : n \in \mathbf{Z} \}$.

9. Show that $\{ x : \cot x = k \} = \{ n\pi + \cot^{-1} k : n \in \mathbf{Z} \}$, k being an arbitrary (but fixed) non-zero real number.

10. Show that if $|k| \geqslant 1$, then the solution-set of the equation $\sec x = k$ is $\{ 2n\pi \pm \sec^{-1} k : n \in \mathbf{Z} \}$.

11. Show that if $|k| \geqslant 1$, then $\csc x = k \Longleftrightarrow x \in \{ n\pi + (-1)^n \csc^{-1} k : n \in \mathbf{Z} \}$.

12. Let $k > 1$ and let S denote the solution-set of the equation $\sin x = k$. Which of the following statements is true :
    A. $S = \phi$.
    B. $S = \{n\pi + (-1)^n \sin^{-1} k : n \in \mathbf{Z} \}$.
    C. $S = \{n\pi + (-1)^n \sin^{-1} (1/k) : n \in \mathbf{Z} \}$.
    D. None of the statements A, B and C is true.

13. Let $k > 1$ and let S denote the solution-set of the equation $\cos x = k$. Which of the following statements is true :
    A. $S = \phi$.
    B. $S = \{2n\pi \pm \cos^{-1} k : n \in \mathbf{Z} \}$.
    C. $S = \{2n\pi \pm \cos^{-1} (1/k) : n \in \mathbf{Z} \}$.
    D. None of the statements A, B and C is true.

14. Find the solution-set of the following equations :
    (i) $\cos x = 1/2 \ \wedge \ x \in [-2\pi, 2\pi]$.
    (ii) $\tan x = 1 \ \wedge \ x \in [0, 4\pi]$.
    (iii) $2 \sin x = -\sqrt{3} \ \wedge \ x \in [-3\pi, 3\pi]$.
    (iv) $\sec x + 2 = 0 \ \wedge \ x \in [-4\pi, 0]$.

9.7. Equations Involving only one Circular Function

Example 2. Solve the equation $\sin^2 x = 1/4$.

Solution.
$$\begin{aligned} \sin^2 x = 1/4 &\Longleftrightarrow (1/2)(1 - \cos 2x) = 1/4, \\ &\Longleftrightarrow 2(1-\cos 2x) = 1, \\ &\Longleftrightarrow \cos 2x = 1/2, \\ &\Longleftrightarrow 2x \in \{2n\pi \pm \pi/3 : n \in \mathbf{Z}\}, \\ &\Longleftrightarrow x \in \{n\pi \pm \pi/6 : n \in \mathbf{Z}\}. \end{aligned}$$

Hence the solution-set of the given equation is
$\{ n\pi \pm \pi/6 : n \in \mathbf{Z}\}$.

Aliter :

$$\begin{aligned}\sin^2 x = \tfrac{1}{4} &\Longleftrightarrow \sin x = 1/2 \vee \sin x = -1/2,\\ &\Longleftrightarrow x \in \{n\pi + (-1)^n \pi/6 : n \in \mathbf{Z}\} \vee x \in \{n\pi + (-1)^n (\pi/6) : n \in \mathbf{Z}\},\\ &\Longleftrightarrow x \in \{n\pi + (-1)^n \pi/6 : n \in \mathbf{Z}\} \vee x \in \{n\pi + (-1)^{n+1} \pi/6 : n \in \mathbf{Z}\},\\ &\Longleftrightarrow x \in \{n\pi \pm \pi/6 : n \in \mathbf{Z}\}. \qquad \text{......(i)}\end{aligned}$$

$$\begin{aligned}\text{Also, } x \in \{ n\pi \pm \pi/6 : n \in \mathbf{Z} \} &\Rightarrow \sin^2 x = \sin^2 (n\pi \pm \pi/6),\\ &\Rightarrow \sin^2 x = \{ (-1)^n \sin (\pm\pi/6)\}^2,\\ &\Rightarrow \sin^2 x = 1/4. \qquad \text{......(ii)}\end{aligned}$$

From (i) and (ii) we find that the solution set of the given equation is $\{ n\pi \pm \pi/6 : n \in \mathbf{Z} \}$.

Example 3. Solve the equation

$$2 \cos^2 x - 5 \cos x - 3 = 0.$$

Solution. Factorizing the left hand member of the given equation we have

$$(2 \cos x + 1) (\cos x - 3) = 0.$$

By setting each factor to zero, we have

$$2 \cos x + 1 = 0, \text{ or } \cos x - 3 = 0$$

i.e., $\cos x = -1/2$, or $\cos x = 3$.

The second of these equations has no solution because the cosine of a number cannot exceed unity. Also $\cos x = -1/2 \Rightarrow x \in \{2n\pi \pm 2\pi/3 : n \in \mathbf{Z}\}$. Hence the required solution set is $\{ 2n\pi \pm 2\pi/3 : n \in \mathbf{Z}\}$.

Check. $x \in \{ 2n\pi \pm 2\pi/3 : n \in \mathbf{Z}\} \Rightarrow 2 \cos^2 x - 5 \cos x - 3 = 2 \cos^2 (2 n\pi \pm 2\pi/3)$

$$\begin{aligned}&- 5 \cos (2n\pi \pm 2\pi/3) - 3,\\ &= 2 \cos^2 2\pi/3 - 5 \cos 2\pi/3 - 3,\\ &= 2 (- \tfrac{1}{2})^2 - 5 (- \tfrac{1}{2}) - 3 = 0.\end{aligned}$$

Example 4. Solve the equation :

$$\sec^2 x - 4 \tan x + 2 = 0.$$

Solution. $\sec^2 x - 4 \tan x + 2 = 0$

$$\begin{aligned}&\Longleftrightarrow \tan^2 x - 4 \tan x + 3 = 0,\\ &\Longleftrightarrow (\tan x - 1) (\tan x - 3) = 0,\\ &\Longleftrightarrow \tan x = 1 \vee \tan x = 3,\\ &\Longleftrightarrow x \in \{n\pi + \pi/4 : n \in \mathbf{Z}\} \vee x \in \{n\pi + \tan^{-1} 3 : n \in \mathbf{Z}\},\\ &\Longleftrightarrow x \in [\{n\pi + \pi/4 : n \in \mathbf{Z}\}] \cup \{n\pi + \tan^{-}3 : n \in \mathbf{Z}\}].\end{aligned}$$

Hence the solution-set of the given equation is

$$\{n\pi + \pi/4 : n \in \mathbf{Z}\} \cup \{n\pi + \tan^{-1} 3 : n \in \mathbf{Z}\}$$

EXERCISES

Solve the following equations. Check the solution in each case.

1. $\cos^2 x = \frac{1}{4}$.
2. $4 \sin^2 x - 3 = 0$.

3. $3 \tan^2 x = 1.$
4. $7 \sin^2 x + 3 \cos^2 x = 4.$
5. $\tan^2 x = 3 \csc^2 x - 1.$
6. $2 \cot^2 x = \csc^2 x.$
7. $4 \sin^2 x + 4 \sin x + 1 = 0.$
8. $2 \sin^2 x - 3 \sin x + 1 = 0.$
9. $2 \cos^2 x + 3 \cos x + 1 = 0.$
10. $\tan^2 x + \tan x - 2 = 0.$
11. $\cot^2 x + 2 \cot x - 3 = 0.$
12. $\csc^2 x - \csc x - 6 = 0.$
13. $\sec^2 x + 3 \sec x - 4 = 0.$
14. $4 \cos^2 x - 4 \sin x = 1.$
15. $\sin^2 x - 2 \cos x + \frac{1}{4} = 0.$
16. $2 (\cos x + \sec x) = 5.$
17. $3 \text{ ton } x + \cot x = 5 \csc x.$
18. $\cos 3x + 8 \cos^3 x = 0.$
19. $\csc^3 2x = 4 \csc 2x.$
20. $2 \sin^2 x/2 = 1 - \cos^2 x.$

9.8. Equations Involving Trigonometric Functions of two or more Multiples of the Variable

Example 4. Solve the equation

$$\cos p x = \cos q x, \ (p \neq q).$$

Solution. $\cos p x = \cos q x$

$$\begin{aligned}
&\iff \cos p x - \cos q x = 0,\\
&\iff 2 \sin \{\tfrac{1}{2} (p + q) x\} \sin \{\tfrac{1}{2} (q - p) x\} = 0,\\
&\iff \sin \{\tfrac{1}{2} (p + q) x\} = 0 \vee \sin \{\tfrac{1}{2} (p + q) x\} = 0,\\
&\iff \tfrac{1}{2} (p + q) x \in \{n\pi : n \in \mathbf{Z}\} \vee \tfrac{1}{2} (q - p) x \in \{n\pi : n \in \mathbf{Z}\},\\
&\iff x \in \left\{\frac{2n\pi}{p+q} : n \in \mathbf{Z}\right\} \vee x \in \left\{\frac{2n\pi}{q-p} : n \in \mathbf{Z}\right\}\\
&\iff x \in \left\{\frac{2n\pi}{q+p} : n \in \mathbf{Z}\right\}.
\end{aligned}$$

Hence the solution-set of the given equation is

$$\left\{\frac{2n\pi}{q \pm p} : n \in \mathbf{Z}\right\}.$$

Example 5. Solve the equation :

$$\cos x + \cos 3 x + \cos 5 x + \cos 7 x = 0.$$

Solution.
$$\begin{aligned}
\cos x + \cos 3x + \cos 5x + \cos 7x &= 2 \cos 2x \cos x + 2 \cos 6 x \cos x,\\
&= 2 \cos x (\cos 2x + \cos 6x),\\
&= 2 \cos x \cos 4 x \cos 2x.
\end{aligned}$$

Therefore the given equation is equivalent to the equation

$$2 \cos x \cos 4 x \cos 2 x = 0,$$

which in turn is equivalent to

$$\cos x = 0 \vee \cos 4 x = 0 \vee \cos 2 x = 0.$$

Now $\{ x : \cos x = 0 \} = \{(2n+1)\, \pi/2 : n \in \mathbf{Z}\}$,

$$\{ x : \cos 4x = 0 \} = \{x : 4x = (2n + 1)\, \pi/2,\ n \in \mathbf{Z}\} = \{(2n + 1)\pi/8 : n \in \mathbf{Z}\},$$
$$\{ x : \cos 2x = 0 \} = \{x : 2x = (2n + 1)\, \pi/2,\ n \in \mathbf{Z}\} = \{(2n + 1)\, \pi/4 : n \in \mathbf{Z}\}.$$

Therefore the solution-set is

$$\{(2n + 1)\, \pi/2 : n \in \mathbf{Z}\} \cup \{(2n + 1)\, \pi/4 : n \in \mathbf{Z}\} \cup \{(2n + 1)\, \pi/8\ n \in \mathbf{Z}\}.$$

EXERCISES

Solve the following equations :

1. $\cos 4x = \cos 6x.$
2. $\sin 3x = \sin 5x.$
3. $\cos 8x + \cos 10x = 0.$
4. $\sin 2x - 6x = 0.$
5. $\tan 4x = \tan 3x.$
6. $\tan 2x = \cot 4x.$
7. $\cos 2x + \sin 3x = 0.$
8. $\cos px + \cos q\, x = 0.$
9. $\sin px + \sin qx = 0.$
10. $\tan m\, x = \cot n\, x.$
11. $\cos x + \cos 3x = \cos 2x.$
12. $\cos 4x - \cos 7x = \cos x.$
13. $\sin 7x = \sin x + \sin 3x.$
14. $\sin 4x - \sin 2x = \cos 3x.$
15. $\cos x + \cos 2x + \cos 3x = 0.$
16. $\sin x + \sin 3x + \sin 5x = 0.$
17. $\sin x + \sin 2x + \sin 3x + \sin 4x = 0.$
18. $\cos 3x = \cos 5x \wedge x \in [-2\pi, 2\pi].$
19. $\cos x + \cos 3x + \cos 5x = 0 \wedge x \in\,]-\pi, \pi[.$
20. $\sin 4x + \sin 2x = \sin 3x \wedge x \in [0, 2\pi].$

9.9. To Solve the Equation

$$a \cos x + b \sin x = c.$$

Let us determine a positive real number r and a real number ϕ in $[0, 2\pi]$ such that

$$a = r \cos \phi,\ b = r \sin \phi. \qquad \text{.........(i)}$$

Squaring both sides of each of the above equations and adding we have,

$$a^2 + b^2 = r^2 \cos^2 \phi + r^2 \sin^2 \phi = r^2.$$

Since we wish r to be positive, therefore we have

$$r = \sqrt{a^2+b^2}. \qquad \text{.........(ii)}$$

Substituting the above value of r in (i) we have

$$\cos \phi = \frac{a}{\sqrt{(a^2+b^2)}},\ \sin \phi = \frac{b}{\sqrt{(a^2+b^2)}}. \qquad \text{.........(iii)}$$

Equations (iii) determine a unique number ϕ between 0 and 2π.

Having determined r and ϕ, let us put $a = r \cos \phi$ and $b = r \sin \phi$ in (i). The given equation then reduces to

$$r \cos \phi \cdot \cos x + r \sin \phi \cdot \sin x = c,$$

or $r \cos (x - \phi) = c$,

or $\cos (x - \phi) = \dfrac{c}{\sqrt{(a^2+b^2)}}$. ....(iv)

Two different cases arise according as $\left| \dfrac{c}{\sqrt{(a^2+b^2)}} \right| > 1$ or

$$\left| \frac{c}{\sqrt{(a^2+b^2)}} \right| \leqslant 1.$$

If $\left| \dfrac{c}{\sqrt{(a^2+b^2)}} \right| > 1$, then the solution-set of (iv) is empty. Since the given equation is equivalent to (iv), therefore the solution-set of the given equation is also empty.

If $\left| \dfrac{c}{\sqrt{(a^2+b^2)}} \right| < 1$, then the solution-set of (iv) is

$$S = \{x : x - \phi = 2\, n\pi \pm \cos^{-1}\left(\frac{c}{\sqrt{(a^2+b^2)}}\right), n\epsilon \mathbf{Z}\},$$

i.e., $S = \{x : x = 2n\pi \pm \cos^{-1}\left(\dfrac{c}{\sqrt{(a^2+b^2)}}\right) + \phi,\ n\epsilon \mathbf{Z}\}$. ....(v)

Since the given equation is equivalent to (iv), therefore the solution-set of the given equation is also S.

Example 7. Solve the equation

$$\cos x + \sqrt{3} \sin x = 1.$$

Solution. Let us determine a positive real number r and a real number ϕ in $[0, 2\pi]$ such that

$1 = r \cos \phi,\ \sqrt{3} = r \sin \phi$. ......(i)

By squaring both sides of each of the equations (i) and adding, we have

$1^2 + (\sqrt{3})^2 = r^2 \cos^2 \phi + r^2 \sin^2 \phi$,

or $4 = r^2$.

Since we wish r to be positive, therefore we have

$r = 2$. ....(ii)

Substituting the above value of r in (i) we have

$\cos \phi = 1/2$, $\sin \phi = \sqrt{3}/2$. ......(iii)

The first of these equations has solutions $\pi/3$, $2\pi - \pi/3$ in $[0, 2\pi]$, while the second equation has solutions $\pi/3$, $\pi - \pi/3$ in $[0, 2\pi]$.

Therefore the equations (iii) have a unique common solution $\phi = \pi/3$ in $[0, 2\pi]$.

We can now write the given equation as

$r \cos \phi \cos x + r \sin \phi \sin x = 1$,

or $r \cos (x - \phi) = 1$,

or $2 \cos (x-\pi/3) = 1$,

or $\cos (x-\pi/3) = 1/2$. ......(iv

The solution-set of equation (iv), and therefore of the given equation (which is equivalent to (iv) is,

$$\{x : x-\pi/3=2n \pi \pm \cos^{-1} 1/2,\ n \in \mathbf{Z}\},$$

i. e., $\{x : x = 2n\pi \pm \pi/3 + \pi/3,\ n \in \mathbf{Z}\}$.

i. e., $\{2n \pi : n \in \mathbf{Z}\} \cup \{2n \pi+2\pi/3 : n \in \mathbf{Z}\}$.

Check.

$x = 2n \pi \Rightarrow \cos x + \sqrt{3} \sin x = \cos 2n \pi + \sqrt{3} \sin 2n \pi = 1+ \sqrt{3}.\ 0=1.$

$$\begin{aligned} x = 2n \pi+2\pi/3 \Rightarrow \cos x + \sqrt{3} \sin x &= \cos (2n\pi+2\pi/3)+\sqrt{3} \sin (2n \pi+2\pi/3), \\ &=\cos 2\pi/3+ \sqrt{3} \sin 2\pi/3, \\ &=-1/2+ \sqrt{3} (\sqrt{3}/2)=1. \end{aligned}$$

Aliter : We may rewrite the given equation in the form

$\sqrt{3} \sin x = 1-\cos x.$

Squaring both sides of (i), we have

$3 \sin^2 x=(1-\cos x)^2$,

or $3 (1-\cos^2 x) = 1-2 \cos x + \cos^2 x$,

or $4 \cos^2 x-2 \cos x -2=0$,

or $(2 \cos x + 1) (\cos x -1)=0$. ......(ii)

Equating each factor of the left hand member to zero, we have

$2 \cos x + 1 =0 \lor \cos x -1=0$,

i.e., $\cos x = -\frac{1}{2} \lor \cos x = 1$.

The solution-set of (ii) is therefore

$S=\{2n \pi +2\pi/3 : n \in \mathbf{Z}\} \cup \{2n \pi : n \in \mathbf{Z}\}$. ......(iii)

S is the solution-set of the equation (ii) which is obtained by squaring the members of (i). Since squaring is an irreversible process, we must check up as to which members of S are roots of the given equation.

When $x=2n \pi+2\pi/3$, $\cos x+ \sqrt{3} \sin x =\cos (2n\pi+2\pi/3)+\sqrt{3} \sin (2n \pi +2\pi/3)=1$.

$$\begin{aligned} \text{When } x =2n \pi -2\pi/3,\ \cos x+ \sqrt{3} \sin x &=\cos (2n \pi-2\pi/3)+\sqrt{3} \sin (2n \pi-2\pi/3), \\ &=\cos (-2\pi/3) + \sqrt{3} \sin (-2\pi/3), \\ &=-1/2+ \sqrt{3} (- \sqrt{3}/2), \\ &=-2. \end{aligned}$$

$$\begin{aligned} \text{When } x = 2n \pi,\ \cos x +\sqrt{3} \sin x &= \cos (2n \pi) +\sqrt{3} \sin (2n \pi), \\ &= 1. \end{aligned}$$

Hence we see that while every member of $\{ 2n \pi +2\pi/3 : n \in \mathbf{Z} \} \cup \{ 2n \pi : n \in \mathbf{Z} \}$ is a solution of the given equation, no member of $\{2n \pi-2\pi/3 : n \in \mathbf{Z} \}$ is a solution.

Thus the solution-set of the given equation is
$\{2n\pi+2\pi/3 : n \in \mathbf{Z}\} \cup \{2n\pi : n \in \mathbf{Z}\}$.

Remark 1. Each member of $\{2n\pi-2\pi/3 : n \in \mathbf{Z}\}$ is a solution of the equation $-\sqrt{3}\sin x = 1-\cos x$. In fact, the solution-set of this equation can be shown to be $\{2n\pi : n \in \mathbf{Z}\} \cup \{2n\pi-2\pi/3 : n \in \mathbf{Z}\}$.

Remark 2. While we have worked out the problem in the above example in a rather detailed manner, the solution can generally be written in a more compact form as in the following example.

Example 8. Solve the equation $2\cos x - 3\sin x = -2$.

Solution. Dividing both sides of the given equation by $\sqrt{(2^2+(-3)^2)}$, we have

$$\frac{2}{\sqrt{13}}\cos x - \frac{3}{\sqrt{13}}\sin x = -\frac{2}{\sqrt{13}}. \qquad \text{.........(i)}$$

Let $\cos^{-1}\frac{2}{\sqrt{13}} = \phi$. Then $0 < \phi < \frac{\pi}{2}$, and $\cos\phi = \frac{2}{\sqrt{13}}$, so that $\sin\phi = \frac{3}{\sqrt{13}}$. We may, therefore, re-write (i) as

$$\cos x\cos\phi - \sin x\sin\phi = -\frac{2}{\sqrt{13}},$$

$$\text{i.e., } \cos(x+\phi) = -\frac{2}{\sqrt{13}}. \qquad \text{.........(ii)}$$

The solution set of (ii) and therefore of the given equation is

$$\{x : x+\phi = 2n\pi \pm \cos^{-1}(-\frac{2}{\sqrt{13}}), n \in \mathbf{Z}\},$$

$$\text{i. e., } \{x : x = 2n\pi \pm \cos^{-1}(-\frac{2}{\sqrt{13}}) - \phi, n \in \mathbf{Z}\},$$

$$\text{i e., } \{2n\pi \pm \cos^{-1}(-\frac{2}{\sqrt{13}}) - \cos^{-1}\frac{2}{\sqrt{13}} : n \in \mathbf{Z}\}.$$

Example 9. Find the solution-set of the equation
$3\cos x + 4\sin x = 10$.

Solution. Dividing both sides of the given equation by $\sqrt{(3^2+4^2)}$, we have

$$\frac{3}{\sqrt{(3^2+4^2)}}\cos x + \frac{4}{\sqrt{(3^2+4^2)}}\sin x = \frac{10}{\sqrt{(3^2+4^2)}},$$

$$\text{i.e., } 3/5\cos x + 4/5\sin x = 2. \qquad \text{.........(i)}$$

Let $\cos^{-1} 3/5 = \phi$. Then $0 < \phi < \pi/2$ and $\cos\phi = 3/5$, so that $\sin\phi = 4/5$. We may, therefore, re-write (i) as

$$\cos x\cos\phi + \sin x\sin\phi = 2, \qquad \text{.........(ii)}$$

$$\text{i.e., } \cos(x-\phi) = 2.$$

Since $|\cos y|$ can never exceed 1, therefore the solution-set of (ii) is empty. Also, since the given equation is equivalent to (ii), therefore the solution-set the given equation is empty.

EXERCISES

1. Solve, by the method of Art. 9.9, the equations in the following problems. Check the solution in each case.

(i) $\sqrt{3}\cos x + \sin x = \sqrt{2}$.
(ii) $\cos x - \sqrt{3}\sin x = 1$.
(iii) $\cos x + \sin x = 1$.
(iv) $\sin x - \cos x = \sqrt{2}$.
(v) $\sqrt{3}\sin x - \cos x = \sqrt{2}$.
(vi) $\cos x + \sqrt{3}\sin x = -1$.

2. Solve, by the method of squaring, the equations in the following probelms. Check the solution in each case.

(i) $\sqrt{3}\cos x - \sin x = \sqrt{2}$.
(ii) $\cos x = \sin x - \sqrt{2}$.
(iii) $\sin x + 1 = \sqrt{3}\cos x$.
(iv) $\cos x - \sin x = \sqrt{2}$.

3. Solve the following equations :

(i) $\cos x + \sin x = \sqrt{2}$.
(ii) $2\cos x - \sin x = 2$.
(iii) $\cos x + 2\sin x = 3$.
(iv) $\sqrt{3}\cos x + \sin x = 1$.
(v) $\tan x + \sec x = \sqrt{3}$.
(vi) $\sqrt{2}\sec x + \tan x = 1$.
(vii) $\csc x = 1 + \cot x$.
(viii) $\csc x = \cot x + \sqrt{3}$.

9.10. Solutions of pairs of equations

Example 10. Find the solution-set of the pair of equations $\cos x = 1/2$ and $\sin x = -\sqrt{3}/2$.

Solution. We shall first find all those x in $[0, 2\pi]$ that satisfy both the given equations and then determine the solution-set by using the fact that the sine and cosine functions are periodic and have a common period 2π.

Now $\{x : \cos x = 1/2 \wedge 0 \leqslant x \leqslant 2\pi\} = \{\pi/3, 5\pi/3\}$,

$\{x : \sin x = -\sqrt{3}/2 \wedge 0 \leqslant x \leqslant 2\pi\} = \{4\pi/3, 5\pi/3\}$.

Therefore

$$\begin{aligned} &\{x : \cos x = 1/2 \wedge \sin x = -\sqrt{3}/2 \wedge 0 \leqslant x \leqslant 2\pi\} \\ &= \{x : \cos x = 1/2 \wedge 0 \leqslant x \leqslant 2\pi\} \wedge \{x : \sin x = \sqrt{3}/2 \wedge 0 \leqslant x \leqslant 2\pi\}, \\ &= \{\pi/3, 5\pi/3\} \cap \{4\pi/3, 5\pi/3\}, \\ &= 5\pi/3 . \end{aligned}$$

By periodicity of the sine and cosine functions,

$$\{x : \cos x = 1/2 \wedge \sin x = -\sqrt{3}/2\} = \{2n\pi + 5\pi/3 : n \in \mathbf{Z}\}.$$

Hence the desired solution set is $\{2n\pi + 5\pi/3 : n \in \mathbf{Z}\}$.

Check. : $x = 2n\pi + 5\pi/3 \ (n \in \mathbf{Z}) \Rightarrow \cos x = \cos(2n\pi + 5\pi/3) = \cos 5\pi/3 = \frac{1}{2}$,
and $\sin x = \sin(2n\pi + 5\pi/3) = \sin 5\pi/3 = -\sqrt{3}/2$.

Alternatively, we could proceed as follows also :

$$\{x : \cos x = 1/2\} = \{2n\pi \pm \pi/3 : n \in \mathbf{Z}\},$$
$$\{x : \sin x = -\sqrt{3}/2\} = \{n\pi + (-1)^n(-\pi/3)\ n \in \mathbf{Z}\}.$$

Now $\{x : \cos x = 1/2 \wedge \sin x = -\sqrt{3}/2\}$

$$= \{x : \cos x = 1/2\} \cap \{x : \sin x = -\sqrt{3}/2\},$$
$$= \{x : x = 2n\pi \pm \pi/3,\ n \in \mathbf{Z}\} \cap \{x : x = n\pi + (-1)^{n+1}\pi/3,\ n \in \mathbf{Z}\},$$
$$= \{x : x = 2n\pi \pm \pi/3,\ n \in \mathbf{Z}\} \cap \{x : x = 2m\pi - \pi/3 \vee x = 2m\pi + 5\pi/3,\ m \in \mathbf{Z}\}.$$
$$= \{x : x = 2n\pi - \pi/3,\ n \in \mathbf{Z}\},$$
$$= \{2n\pi - \pi/3 : n \in \mathbf{Z}\}.$$

Remark. By the first method we had obtained the solution-set as $\{2n\pi + 5\pi/3 : n \in \mathbf{Z}\}$, whereas we have now obtained the solution-set as $\{2n\pi - \pi/3 : n \in \mathbf{Z}\}$ It *appears* as if we have obtained two distinct solution-sets for the same pair of equations. (This, we know, cannot happen). However, we can easily verify that $\{2n\pi + 5\pi/3 : n \in \mathbf{Z}\} = \{2n\pi - \pi/3 : n \in \mathbf{Z}\}$. Such a situation might arise (in fact, it might have already arisen !) in many places. In such cases we should always try to examine whether two apparently distinct sets are in fact distinct or are simply two different ways of expressing the same set.

Example 11. Find the solution-set of the pair of equations $\cos(2x + 3y) = 1/2$, $\cos(3x + 2y) = \sqrt{3}/2$.

Solution. $\cos(2x + 3y) = 1/2 \Longleftrightarrow 2x + 3y = 2p\pi \pm \pi/3,\ p \in \mathbf{Z}$ ....(i)
$\cos(3x + 2y) = \sqrt{3}/2 \Longleftrightarrow 3x + 2y = 2q\pi \pm \pi/6,\ q \in \mathbf{Z}$ ....(ii)

From (i) and (ii) we find that we have to solve the equations

$$2x + 3y = 2p\pi \pm \pi/3,\ p \in \mathbf{Z} \qquad \text{.........(iii)}$$
$$3x + 2y = 2q\pi \pm \pi/6,\ q \in \mathbf{Z} \qquad \text{.........(iv)}$$

Solving the above equations we have

$$x = \tfrac{3}{5}(2q\pi \pm \pi/6) - \tfrac{2}{5}(2p\pi \pm \pi/3),\ p, q \in \mathbf{Z},$$
$$y = \tfrac{3}{5}(2p\pi \pm \pi/3) - \tfrac{2}{5}(2q\pi \pm \pi/6),\ p, q \in \mathbf{Z}.$$

Hence the required solution-set is
$\{(x, y) : x = \frac{3}{5}(2q\pi \pm \pi/6) - \frac{2}{5}(2p\pi \pm \pi/3)\ ;\ y = \frac{3}{5}(2p\pi \pm \pi/3) - \frac{2}{5}(2q\pi \pm \pi/6)\ ;\ p, q \in \mathbf{Z}\}$.

EXERCISES

1. Find the solution-set of each of the following pairs of equations in the following problems ;

(i) $\cos x = -1/2$, $\sin x = \sqrt{3}/2$.
(ii) $\cos x = -\sqrt{3}/2$, $\sin x = 1/2$.
(iii) $\cos x = 1/\sqrt{2}$, $\sin x = -1/\sqrt{2}$.
(iv) $\cos x = 1/2$, $\sin x = -1/2$.
(v) $\sin x = \sqrt{3}/2$, $\tan x = -\sqrt{3}$.
(vi) $\sin x = -1/2$, $\tan x = -1/\sqrt{3}$.
(vii) $\cos x = 1/2$, $\tan x = -\sqrt{3}$.

2. Find the set of all those x in $[-2\pi, 2\pi]$ for which $\cos x = -1/2$ and $\sin x = -\sqrt{3}/2$.

3. Find the solution-set of each of the following pairs of equations in the following problems :

(i) $\cos(x+y) = 1/\sqrt{2}$, $\cos(x-y) = 1/2$.
(ii) $\cos(2x+y) = 1/2$, $\cos(x+2y) = \sqrt{3}/2$.
(iii) $\sin(x+y) = 1/\sqrt{2}$, $\cos(2x-y) = -\sqrt{3}/2$.
(iv) $\cos(x+3y) = 1/2$, $\tan(3x-y) = 1/\sqrt{3}$.
(v) Find the set of all those (x, y) for which
$-\pi/2 \leqslant x \leqslant \pi/2, -\pi/2 \leqslant y \leqslant \pi/2$, $\cos(x-y) = \sqrt{3}/2$, $\sin(x+y) = 1/\sqrt{2}$.

9.11. Trigonometric Inequations

So far we have been dealing with trigonometric equations. We shall now consider trigonometric inequations. The method of determining solution-sets of trigonometric inequations will become clear from the following examples.

Example 12. Find the solution-set of the inequation $\sin x \leqslant 1/\sqrt{2}$.

Solution. In the interval $[-\pi/2, \pi/2]$ x increases strictly from -1 to 1 and assumes each value only once. In particular, it assumes the value $1/\sqrt{2}$ at $x = \pi/4$ Therefore the solution-set of $\sin x \leqslant 1/\sqrt{2}$ in $[-\pi/2, \pi/2]$ is $\{x : -\pi/2 \leqslant x \leqslant \pi/4\}$.

In the interval $[\pi/2, 3\pi/2]$, $\sin x$ decreases strictly from 1 to -1 and assumes each value only once. In particular, it assumes the value $1/\sqrt{2}$ at $x = 3\pi/4$. Therefore the solution set of $\sin x \leqslant 1/\sqrt{2}$ in $[\pi/2, 3\pi/2]$ is $\{y : 3\pi/4 \leqslant y \leqslant 3\pi/2\}$. Therefore the solution-set of the given inequation in

$[-\pi/2, 3\pi/2]$ is $S = \{y : -\pi/2 \leqslant y \leqslant \pi/4\} \cup \{y : 3\pi/4 \leqslant y \leqslant 3\pi/2\}$.

By periodicity of $\sin x$, the solution-set is

$\{2n\pi + y : y \in S, n \in \mathbf{Z}\}$,

i. e., $\{2n\pi + y : -\pi/2 \leqslant y \leqslant \pi/4, n \in \mathbf{Z}\} \cup \{2n\pi + y : 3\pi/4 \leqslant y \leqslant 3\pi/2\}$.

Example 13. Find the solution-set of the inequation $\cos^2 x < 1/4$.

Solution. $\cos^2 x < 1/4 \Longleftrightarrow -1/2 < \cos x < 1/2$.

In the interval $[0, \pi]$, cos x decreases strictly from 1 to -1 and assumes each value only once. In particular, it assumes the value $1/2$ at $x = \pi/3$ and $-1/2$ at $x = 2\pi/3$.

Therefore the solution-set of $-1/2 < \cos x < 1/2$ in $[0, \pi]$ is $\{y : \pi/3 < y < 2\pi/3\}$.

In the interval $[\pi, 2\pi]$ cos x increases strictly from -1 and 1 and assumes each value only once. In particular, it assumes the value $-1/2$ at $x = 4\pi/3$ and $1/2$ at $x = 5\pi/3$.

Therefore the solution-set of the given inequation in $[0, 2\pi]$ is $S = \{y : \pi/3 < y < 2\pi/3\} \cup \{y : 4\pi/3 < y < 5\pi/3\}$.

By periodicity of cos x, the solution-set is

$\{2n\pi + y : y \in S, n \in \mathbf{Z}\}$

i. e., $\{2n\pi + y : \pi/3 \leqslant y < 2\pi/3, n \in \mathbf{Z}\} \cup \{2n\pi + y : 4\pi/3 < y < 5\pi/3\}$.

Example 14. Find the solution-set of the inequation $\tan^2 x > 2$.

Solution. $\tan^2 x > 2 \Longleftrightarrow \tan x > \sqrt{2} \vee \tan x < -\sqrt{2}$.

Let $\alpha = \tan^{-1}\sqrt{2}$ so that $-\alpha = \tan^{-1}(-\sqrt{2})$. In the interval $]-\pi/2, \pi/2[$ tan increases strictly and assumes each value exactly once. In particular, it assumes the value $-\sqrt{2}$ only at α and the value $\sqrt{2}$ only at α. Therefore the solution set of the given inequation is $]-\pi/2, \pi/2[$ is $S = \{y : -\pi/2 < y < -\alpha\} \cup \{y : \alpha < y < \pi/2\}$

By periodicity of tan x, the solution-set is

$\{n\pi + y : y \in S, n \in \mathbf{Z}\}$,

i. e. $\{n\pi + y : -\pi/2 < y < -\tan^{-1}\sqrt{2}, n \in \mathbf{Z}\} \cup \{n\pi + y : \tan^{-1}\sqrt{2} < y < \pi/2, n \in \mathbf{Z}\}$.

EXERCISES

1. For each of the inequations in problems (i) to (iv), find the set of all those x in $[-\pi/2, 3/\pi 2]$ that satisfy it :

(i) $\sin x \leqslant 1/2$.
(ii) $\sin^2 x < 3/4$.
(ii) $\csc x \leqslant \sqrt{2}$.
(iv) $\csc^2 x < 2$.

2. For each of the inequations in the following problems, find the set of all those x in $[0, 2\pi]$ that satisfy it :

(i) $\cos x < 1/2$.
(ii) $\cos^2 x > 1/4$.
(iii) $\cos x + \sqrt{3} \sin x \leqslant \sqrt{2}$.
(iv) $\cos x - \sqrt{3} \sin x \geqslant 1$.

3. For each of the inequations in the following problems, find the set of all those x in $]\pi/2, \pi/2$ that satisfy it :

(i) $-1 \leqslant \tan x \leqslant 1$.
(ii) $|\tan x| < 1\sqrt{3}$.
(iii) $\cot^2 x < 16$.
(iv) $\sec^2 x \geqslant 10$.

4. Find the solution-set of each of the inequations in the following problems :
   - (i) $\sin x > 1/\sqrt{2}$.
   - (ii) $|\cot x| \leqslant 1$.
   - (iii) $\cos^2 x \leqslant 1/4$.
   - (iv) $\sin^2 x > 3/4$.
   - (v) $|\tan x| \leqslant 1/3$.
   - (vi) $\cos x + \sin x \leqslant 1$.
   - (vii) $4 \cos x + 3 \sin x \leqslant 2$.
   - (viii) $1/3 \leqslant \sin x < 1/2$.

ASSORTED EXERCISES ON CHAPTER 9

1. Solve the following equations :
   - (i) $2(\cos^4 x + \sin^4 x) = 1$.
   - (ii) $\tan x + \tan(\pi/4 + x) = 2$.
   - (iii) $\tan 3x = \tan 2x + \tan x$.
   - (iv) $2 \sin 2x - \sin x = 0$.
   - (v) $\cot x + \tan x = 2 \csc x$.
   - (vi) $\cot x - \tan x = 2$.
   - (vii) $5 \tan^4 x - 1 = \tan^2 x$.
   - (viii) $\cos x + \sin x = \cos 2x + \sin 2x$.
   - (ix) $1 + \cos x = 2 \sin^2 x$.
   - (x) $2 \tan x + \sec x = 1$.
   - (xi) $2 \sin^4 x - \cos^2 x = 0$.
   - (xii) $\sin x + \csc x = 1$.
   - (xiii) $\tan^2 x - (1 + \sqrt{3}) \tan x + \sqrt{3} = 0$.
   - (xiv) $\sin x + \sin 2x + \sin 3x + \sin 4x = 0$.
2. Find the solution-set of the pair of equations
   $\sin x = \sqrt{3}/2, \quad \cos x = -1/2$.
3. Find the solution-set of the pair of equations
   $\cos(3x - 4y) = 1/2, \; \sin(3x + 4y) = 1/\sqrt{2}$.
4. Find the set of all those (x, y) for which
   $\cos(2x + y) = -1/2, \; \cos(x + 2y) = 1/\sqrt{2}, \; 0 \leqslant x \leqslant 3\pi/2, \; 0 \leqslant y \leqslant 3\pi/2$
5. Find the set of all those x in $[-\pi/2, 3\pi/2]$ that satisfy the inequation $\sin x \leqslant \sqrt{3}/2$.
6. Find the solution-set of the inequation $\tan^2 x < 1/3$.
7. Find the solution-set of the inequation $\cos 2x + 2 \sin 2x < \sqrt{5}/2$.

ANSWERS

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1. (i) 12° 15′ 36″ (ii) 37° 37′ 12″ (iii) 42° 9′ 0″.
2. (i) 19°·29 (ii) 3°·26 (iii) 2°·88.
3. 4th, 2nd, 2nd, 1st, 4th.
4. 75°.
5. 60°, 30°, 45°, 9°.
   114°·58, 171°·86, 286°·44, 17°·19, 572°·88 (correct to two places of decimals).
6. 4·19, 1·05, ·79, ·52, ·31, 1·83 (correct to two places of decimals).
7. 10·47 cm (correct to two place of decimals).
8. 6π or 18·85 radians correct to two places of decimals.
9. one.
10. 8·4.
11. 1131428·57 cm.
12. 5 : 4.
13. 1°·44 or ·025° per second.
14. 30 radius per second.
15. 37775 km.

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3. tan 50°, sin 57°, cos 99°, cot 275°, sec 298°, csc 182°.
4. No.
5. Yes for $a = 1$ and for $a = -1$.
6. No

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1. (i) 1st (ii) 4th (iii) 1st (iv) 2nd (v) 2nd (vi) 3rd (vii) 4th (viii) 3rd
2. (i) True (ii) False (iii) True
3. (i) 3rd $\wedge$ 4th (ii) 1st $\wedge$ 4th (iii) 1st $\wedge$ 3rd (iv) 2nd $\wedge$ 4th (v) 2nd $\wedge$ 3rd (vi) 1st $\wedge$ 2nd.

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1. 1, 0, 0, -1, -1, -1, 1 respectively.
2. (i) 1 (ii) 1 (iii) 1 (iv) 0.
3. tangent and secant.

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1. (i) $\text{Sin } \theta = 12/13$, $\cos \theta = 5/13$, $\tan \theta = 12/5$,
$\sin \phi = 5/13$, $\cos \phi = 12/13$, $\tan \phi = 5/12$.

(ii) $\text{Sin } \theta = 24/25$, $\cos \theta = 7/25$, $\tan \theta = 24/7$,
$\sin \phi = 7/25$, $\cos \phi = 24/25$, $\tan \phi = 7/24$.

(iii) $\text{Sin } \theta = 4/5$, $\cos \theta = 3/5$, $\tan \theta = 4/3$,
$\sin \phi = 3/5$, $\cos \phi = 4/5$, $\tan \phi = 3/4$.

(iv) $\text{Sin } \theta = \frac{b}{\sqrt{a^2+b^2}}$, $\cos \theta = \frac{a}{\sqrt{a^2+b^2}}$, $\tan \theta = \frac{b}{a}$,

$\sin \phi = \frac{a}{\sqrt{a^2+b^2}}$, $\cos \phi = \frac{b}{\sqrt{a^2+b^2}}$, $\tan \phi = \frac{a}{b}$.

2.

Angle	Sin	Cos	Tan	Cot	Sec	Csc
$\pi/6$	$\frac{1}{2}$	$\sqrt{3}/2$	$1/\sqrt{3}$	$\sqrt{3}$	$2/\sqrt{3}$	2
$\pi/4$	$1/\sqrt{2}$	$1/\sqrt{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$	$1/\sqrt{3}$	2	$2/\sqrt{3}$

3. (i) $\frac{\sqrt{6} + \sqrt{2}}{4}$ (ii) $\frac{\sqrt{2} - \sqrt{6}}{4}$ (iii) $2 - \sqrt{3}$ (iv) $2 + \sqrt{3}$.

6. $\sin 300° = -\sqrt{3}/2$, $\cos 300° = \frac{1}{2}$, $\tan 300° = -\sqrt{3}$,
$\csc 300° = -2/\sqrt{3}$, $\sec 300° = 2$, $\cot 300° = -\sqrt{\frac{1}{3}}$.

7. $\sin 5\pi/6 = \frac{1}{2}$, $\cos 5\pi/6 = -\sqrt{3}/2$,
$\tan 5\pi/6 = \sqrt{\frac{1}{3}}$, $\cot 5\pi/6 = -\sqrt{3}$,
$\csc 5\pi/6 = 2$, $\sec 5\pi/6 = -2/\sqrt{3}$.

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2(i) Identity (ii) Conditional equation.

(iii) Identity.

(iv) Identity.

(v) Identity.

4. $\theta = \pi/2, \theta = -\pi/2.$

5. $\theta = \pi, \theta = 0.$

6. $\theta = \pi/3, \quad \pi/6.$

7. $\theta = 2\pi/3, \theta = 5\pi/4.$ [Read 'exercise 2 (iii)' instead of 'exercise 2 (ii)']

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1. (i) $\cos\theta = \pm\, 12/13$, $\tan\theta = \pm\, 5/12$, $\cot\theta = \pm\, 12/5$, $\sec\theta = \pm\, 13/12$, $\csc\theta = 13/5$.
   (ii) $\sin\theta = \pm\, 3/5$, $\tan\theta = \mp\, 3/4$, $\cot\theta = \mp\, 4/3$, $\sec\theta = -\, 5/4$, $\csc\theta = \pm\, 5/3$.
   (iii) $\sin\theta = \pm\, 24/25$, $\cos\theta = \pm\, 7/25$, $\cot\theta = 7/24$, $\csc\theta = \pm\, 25/24$, $\sec\theta = \pm\, 25/7$.
   (iv) $\sin\theta = \pm\, 1$, $\cos\theta = 0$, $\tan\theta$ is not defined, $\csc\theta = \pm\, 1$, $\sec\theta$ is not defined.
   (v) $\sin\theta = \pm\, \sqrt{8}/3$, $\cos\theta = 1/3$, $\tan\theta = \pm\, \sqrt{8}$, $\cot\theta = \pm\, 1/\sqrt{8}$, $\csc\theta = \pm\, 3/\sqrt{8}$.
   (vi) $\sin\theta = -\, 1/4$, $\cos\theta = \pm\, \sqrt{15}/4$, $\tan\theta = \mp\, 1/\sqrt{15}$, $\cot\theta = \mp\, \sqrt{15}$, $\sec\theta = \pm\, 4/\sqrt{15}$.

2. (i) $\sin\theta = 3/5$, $\tan\theta = 3/4$, $\cot\theta = 4/3$, $\sec\theta = 5/4$, $\csc\theta = 5/3$.
   (ii) $\cos\theta = -\, 12/13$, $\tan\theta = -\, 5/12$, $\cot\theta = -\, 12/5$, $\sec\theta = -\, 13/12$, $\csc\theta = 13/5$.
   (iii) $\sin\theta = -\, 15/17$, $\cos\theta = 8/17$, $\cot\theta = -\, 8/15$, $\sec\theta = -\, 15/8$, $\csc\theta = -\, 17/15$.
   (iv) $\sin\theta = 3/5$, $\cos\theta = -\, 4/5$, $\tan\theta = -\, 3/4$, $\csc\theta = 5/3$, $\sec\theta = -\, 5/4$.
   (v) $\sin\theta = -12/13$, $\cos\theta = -5/13$, $\tan\theta = 12/5$, $\cot\theta = 5/12$, $\sec\theta = -13/5$.
   (vi) $\sin\theta = -\, 24/25$, $\cos\theta = 7/25$, $\tan\theta = -\, 24/7$, $\cot\theta = -7/24$, $\csc\theta = -\, 25/24$.
   (vii) $\sin\theta = -\, \sqrt{3}/2$, $\tan\theta = -\, \sqrt{3}$, $\cot\theta = -\, 1/\sqrt{3}$, $\mathrm{sce}\,\theta = 2$, $\csc\theta = -\, 2\sqrt{3}$.
   (viii) $\cos\theta = \sqrt{7}/4$, $\tan\theta = 3/\sqrt{7}$, $\cot\theta = \sqrt{7}/3$, $\sec\theta = 4/\sqrt{7}$, $\csc\theta = 4/3$.
   (ix) $\sin\theta = 3/\sqrt{13}$, $\cos\theta = 2/\sqrt{13}$, $\cot\theta = 2/3$, $\sec = \sqrt{13}/2$, $\csc\theta = \sqrt{13}/3$.
   (x) $\sin\theta = 6/\sqrt{61}$, $\cos\theta = -\, 5/\sqrt{61}$, $\tan\theta = -\, 6/5$, $\sec\theta = -\, \sqrt{61}/5$, $\csc\theta = \sqrt{61}/6$.
   (xi) $\sin\theta = 1/3$, $\cos\theta = -\, \sqrt{8}/3$, $\tan\theta = -\, 1/\sqrt{8}$, $\cot\theta = -\, \sqrt{8}$, $\sec\theta = -\, \sqrt{3}/8$.
   (xii) $\sin\theta = \sqrt{21}/5$, $\cos\theta = -\, 2/5$, $\tan\theta = -\, \sqrt{21}/2$, $\cot\theta = -\, 2/\sqrt{21}$, $\csc\theta = 5/\sqrt{21}$.
   (xiii) $\sin\theta = 12/13$, $\tan\theta = -\, 12/5$, $\cot\theta = -\, 5/12$, $\sec\theta = -\, 13/5$, $\csc\theta = 13/12$.

(xiv) $\cos\theta = -\sqrt{3}/2$, $\tan\theta = 1/\sqrt{3}$, $\cot\theta = \sqrt{3}$, $\sec\theta = -2/\sqrt{3}$, $\csc\theta = -2$.

(xv) $\sin\theta = -2/\sqrt{5}$, $\cos\theta = -1/\sqrt{5}$, $\cot\theta = 1/2$, $\sec\theta = -\sqrt{5}$,
$\csc = -\sqrt{5}/2$.

(xvi) $\sin\theta = -4/5$, $\cos\theta = 3/5$, $\tan\theta = -4/3$, $\sec\theta = 5/3$, $\csc\theta = -5/4$.

(xvii) $\sin\theta = -7/25$, $\cos\theta = -24/25$, $\tan\theta = 7/24$, $\cot\theta = 24/7$,
$\sec\theta = -25/24$.

(xviii) $\sin\theta = -5/13$, $\cos\theta = 12/13$, $\tan\theta = -5/12$, $\cot\theta = -12/5$,
$\csc\theta = -13/5$.

3. -5.

4. (i) 1st or 2nd (ii) 2nd or 3rd (iii) 1st or 3rd (iv) 2nd or 4th (v) 1st or 2nd (vi) 2nd or 4th.

5. (i) True (ii) False.

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1. (i) True (ii) False (iii) True (iv) True (v) True (vi) False.

2. (i) True (ii) False (iii) True (iv) True.

3. $\tan -90^\circ \wedge \sec -90^\circ$ are not defined, all others are defined.

4. (i) $-{\cdot}79$ (approx.) (ii) $-{\cdot}75$ (iii) 0.

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1. $\sin\theta = -7/25$, $\cos\theta = 24/25$, $\tan\theta = -7/24$,
$\csc\theta = -25/7$, $\sec\theta = 25/24$, $\cot\theta = -24/7$.

6. (i) 4th (ii) 4th (iii) 2nd and 4th (iv) 1st and 2nd.

7. Cotangent, cosecant.

8. Sin $330^\circ = -\frac{1}{2}$, $\cos 330^\circ = \sqrt{3}/2$, $\tan 330^\circ = -1/\sqrt{3}$,
$\csc 330^\circ = -2$, $\sec 330^\circ = 2\sqrt{}/3$, $\cot 330^\circ = -\sqrt{3}$.

11. The statement is true.

12. -5.

17. $\sin(270+\theta) = -\cos\theta$; $\cos(270+\theta) = \sin\theta$; $\tan(270+\theta) = -\cot\theta$;
$\csc(270+\theta) = -\sec\theta$; $\sec(270+\theta) = \csc\theta$; $\cot(270+\theta) = -\tan\theta$;
$\sin(270-\theta) = -\cos\theta$; $\cos(270-\theta) = -\sin\theta$; $\tan(270-\theta) = \cot\theta$;
$\csc(270-\theta) = -\sec\theta$; $\sec(270-\theta) = -\csc\theta$; $\cot(270-\theta) = \tan\theta$;

18. $\sin(90-\theta) = \cos\theta$; $\cos(90-\theta) = \sin\theta$; $\tan(90-\theta) = \cot\theta$;
$\cot(90-\theta) = \tan\theta$; $\sec(90-\theta) = \csc\theta$; $\csc(90-\theta) = \sec\theta$.

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6. Sin $(A+B) = 56/62$.
Cos $(A+B) = 63/65$. [Read $\sin B = 5/13$.]

8. (i) $1/2$ (ii) $\sqrt{3}/2$ (iii) $1/\sqrt{2}$ (iv) $1/2$

10. when $qs = pr$ then A and B are complementary.

15. For Q.5 (v) : True for all values of θ and ϕ except when θ and, or, ϕ are odd multiples of $\pi/2$.
For Q 5(vi) : -do-

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1. (i) $\sin 26°$ (ii) $\cos 50°$ (iii) $\tan 2\pi/5$ (iv) $\cos 2\pi/9$ (v) $\sin 62°$ (vi) $\cos 46°$ (vii) $\cos 34°$ (viii) $\sin 45°$ (ix) $\cos 3\pi/10$ (x) $\cot 3\pi/7$ [Read $1 - 3\tan^2 \pi/7$ in the numerator].

2. $2\cos^2 65° - 1$.

3. $4\cos^3 18° - 3\cos 18°$.

4. $\dfrac{\cot^3 24° - 3\cot 24°}{3\cot^2 24° - 1}$.

5. (i) $\sin 2\theta = \dfrac{24}{25}$, $\cos 2\theta = -\dfrac{7}{25}$, $\tan 2\theta = -\dfrac{24}{7}$.

   (ii) $\sin 2\theta = \dfrac{-5}{72}\sqrt{119}$, $\cos 2\theta = \dfrac{-47}{72}$, $\tan 2\theta = \dfrac{5\sqrt{119}}{47}$.

   (iii) $\sin 2\theta = \dfrac{336}{625}$, $\cos 2\theta = \dfrac{-527}{625}$, $\tan 2\theta = \dfrac{-336}{527}$.

   (iv) $\sin 2\theta = \dfrac{-4}{5}$, $\cos 2\theta = \dfrac{3}{5}$, $\tan 2\theta = \dfrac{-4}{3}$.

8. For all values of θ between 0° and 360° except for 30°, 150°, 210° and 330°.

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1. (i) $-\sqrt{3}/2$ (ii) $-1/\sqrt{2}$ (iii) $-1/\sqrt{3}$ (iv) 1 (v) -2 (vi) -2 (vii) 0 (viii) $-\dfrac{3}{2}$.

2. (i) $-\cos\theta$, (ii) $-\sin\theta$ (iii) $-\tan\theta$ (iv) $\cot\theta$ (v) $-\sec\theta$ (vi) $\sec\theta$ (vii) $-\tan^4\theta$.

3. (i) $\cos 19°$ 3(ii) $-\cos 27°$ 3(iii) $\cot 25°$ (iv) $-\cot 42°$ (v) $-\csc 13°$, (vi) $\sec 24°$.

4. 30°, 330°.

5. 240°.

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1. $\sin 15^\circ = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$;
   $\cos 15^\circ = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$;
   $\tan 15^\circ = \dfrac{\sqrt{3}-1}{\sqrt{3}+1}$;
3. $\sin \theta/2 = 1/\sqrt{5}$;
   $\cos \theta/2 = 2/\sqrt{5}$;
   $\tan \theta/2 = 1/2$.
4. $\sin \theta/2 = 4/5$, $\cos \theta/2 = -3/5$, $\tan \theta/2 = -4/3$.
5. $\sin \theta/2 = 5/\sqrt{26}$, $\cos \theta/2 = 1/\sqrt{26}$, $\tan \theta/2 = 5$.
6. $\sin \theta/2 = 1/\sqrt{5}$, $\cos \theta/2 = -2/\sqrt{5}, = \tan \theta/2 = -1/2$.
8. (i) True (ii) True (iii) False; L. H.S. < 0 while R. H.S. > 0 (iv) True (v) True.

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1. (i) $\sin 90^\circ + \sin 30^\circ$ (ii) $\sin 100^\circ - \sin 40^\circ$ (iii) $\cos 60^\circ + \cos 30^\circ$ (iv) $\cos 20^\circ - \cos 100^\circ$ (v) $\sin 6\theta + \sin 4\theta$ (vi) $\frac{1}{2}\cos 2A + \frac{1}{2}\cos 2B$ (vii) $2\sin 2\theta - 2\sin 120^\circ$ (viii) $4\cos 2(A-B) - 4\cos 2(A+B)$.
2. (i) $2\sin 36^\circ \cos 12^\circ$ (ii) $2\cos 42^\circ \sin 30^\circ$ (iii) $2\cos 4\theta \cos\theta$ (iv) $-2\sin 4\pi/9 \sin 2\pi/9$ (v) $2\cos 2\theta \sin\phi$ [Read the question as $\sin(2\theta+\phi) - \sin(2\theta-\phi)$] (vi) $2\sin 3\alpha \sin 5\beta$.
3. (iv) Read $\sin(120^\circ+\theta) + \sin(240^\circ+\theta)$ instead of $\sin(120^\circ+\theta)\sin(240^\circ+\theta)$.

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1. (i) ·4253 (ii) ·6648 (iii) ·6168 (iv) ·6371 (v) 1·241 (vi) 1·167 (vii) ·9644 (viii) ·9605 (ix) 2·229 (x) 1·520.
2. (i) 7° 10′ (ii) 14° 50′ (iii) 58° 30′ (iv) 55° 10′ (v) 67° 20′ (vi) 54° 30′ (vii) 67° 50′ (viii) 69° 20′ (ix) 26° 30′ (x) 59° 40′.

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1. (i) ·4754 (ii) ·6187 (iii) ·8371 (iv) ·8806 (v) 2·5 (vi) 1·020 (vii) ·7300 (viii) ·8313 (ix) 1·517 (x) 1·131 (xi) 1·037 (xii) 1·712.
2. 3° 36′ (ii) 57° 27′ (iii) 23° 24′ (iv) 31° 26′ (v) 66° 47′ (vi) 45° 48′ (vii) 71° 25′ (viii) 66° 6′ (ix) 29° 36′ (x) 67° 24′ (xi) 56° 44′ (xii) 27° 15′.

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1. (i) 30 sq. cm. (ii) 336 sq. cm. (iii) 2430 sq. cm. (iv) $\frac{1}{4}(3 + \sqrt{3})$ sq. m. (v) 30 sq. cm.

2. 5 cm.

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1. (i) $75\sqrt{3}/2$ (ii) $50 \cot 22\frac{1}{2}°$ (iii) $(125/2) \cot 18°$.

2. $5 \cot 18°$, $5 \csc 18°$.

5. $(9/2) \cot 22\frac{1}{2}° - 6\sqrt{3}$ sq. dm.

7. $n = 3$ 8. $n = 6$.

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1. (i) $\log_2 64 = 6$ (ii) $\log_3 81 = 4$ (iii) $\log_7 (1/49) = -2$ (iv) $\log_6 1 = 0$ (v) $\log_{27} 9 = 2/3$ (vi) $\log_{32} 4 = 2/5$.

2. (i) 16 (ii) $\frac{1}{3}$ (iii) 1 (iv) 27 (v) 1/49 (vi) $\frac{1}{8}$ (vii) 2 (viii) 3 (ix) 10 (x) 2 (xi) $a \in \mathbf{R}, a \neq 0, a \neq 1$ (xiii) 4 (xiv) 5 (xv) 3 (xvi) -4 (xvii) $\frac{1}{2}$ (xviii) $\frac{1}{3}$.

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1. (i) $\log 3 + 2 \log 2 + 2 \log 7 + \log 11 + \log 17$.
   (ii) $10 \log 2 + 4 \log 7 - 3 \log 9$.
   (iii) $2 \log x + 3 \log y + 4 \log z - \log p - \log q - \log r$.
   (iv) $\frac{1}{2} [\log(s-b) + \log(s-c) - \log s - \log (s-a)]$.

2. (i) $\log s = -\log 2 + \log g + 2 \log t$.
   (ii) $\log V = 2 \log 2 - \log 3 + \log \pi + 3 \log r$.
   (iii) $\log t = \log 2 + \log \pi + \frac{1}{2} \log 1 - \frac{1}{2} \log g$.
   (iv) $\log F = \log k + \log m_1 + \log m_2 - 2 \log r$.

3. (i) 1·8060 (ii) $-5·2481$ (iii) 1·3801 (iv) ·4034 (v) 1·59305 (vi) ·31225.

5. (i) $x = 3$ (ii) $x = 3$ (iii) $x = 0$ (iv) $x = 16$ (v) $x = 1$ (vi) $x = 190/53$ (vii) $x = -5$ or $-7/5$.

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1. (i) 2 (ii) 1 (iii) 4 (iv) -1 (v) -5 (vi) 4 (vii) 2 (viii) 3 (ix) 1 (x) 0 (xi) -3 (xii) -2.

2. (i) 2 (ii) -4 (iii) -1 (iv) 3 (v) -5 (vi) -7.

3. (i) 673·1 (ii) 6731 (iii) 67·31 (iv) $6731 \cdot 10^{12}$ (v) ·6731 (vi) ·06731.

4. 16.

5. 13.

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1. (i) 3·6407 (ii) 4·4581 (iii) 5·9720 (iv) 0·6664 (v) 1·3955 (vi) 2·0924 (vii) $8·7882^{-10}$ (viii) $5·5051^{-10}$ (ix) $7·6885^{-10}$.

2. (i) 25·96 (ii) 488·8 (iii) 6·551 (iv) ·4403 (v) ·02674 (vi) ·3448 (vii) ·002101 (viii) ·0007472 (ix) ·00009934.

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1. 23·17 2. 80·35 3. 7·30 4. 35·14 5. ·005748 6. ·01813 7. 1·327 8. 76·70 9. ·09198 10. 21·09 11. ·5117 12. ·2705 13. 1·611 14. 89·93 15. 6·176 16. 13·16 17. $-7·067$ 18. $-14·07$ 19. 1·022 20. ·433.

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1. (i) 1·641 (ii) ·7157 (iii) ·7097 (iv) 1·946 (v) ·074 (vi) 1·4753 (vii) 3·83 (viii) ·9778 (ix) 1·845 (x) 1·028 (xi) 1·585, 2·322 (xii) 1·465, ·631 (xiii) ·6825, ·4306 (xiv) ·3562.

2. (i) $x > 2·808$ (ii) $x \leqslant ·7372$ (iii) $x > ·03523$ (iv) $x \geqslant 2·181$ (v) $x > 2·322$ (vi) $x < -·3449$.

3. (i) 5·996 (ii) 3·402.

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1. Rs 8147. 2. Rs 7247. 3. Rs 5099. 4. Rs 657. 5. $3·6 \times 10^{21}$ ergs. 6. $1·125·10^{8}$ k.w.h. [Read 10^{6} instead of 16^{6}] 7. 1421. 8. 12 hrs., 2 min., 24 sec. 9. K. = ·0000969 10. 2536 years 11. (i) positive (ii) negative (iii) $k = ·1003 = ·001505$. 12. $T = 1·269$ ($\pi = 3·1416$). 13. 1244 cubic cm. 14. $P = 606·6$ 15. No, rails should be spaced at least ·8868 cm. apart.

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1. (i) $9·7201-10$ (ii) $9·5704-10$ (iii) 0·0986 (iv) 0·3586 (v) 0·1810 (vi) 0·1082 (vii) $9·6558-10$ (viii) $9·9770-10$ (ix) 0·1438 (x) 0·0834.

2. (i) 15° 30′ (ii) 24° 10′ (iii) 11° 40′ (iv) 22° 4′ (v) 15° 33′ (vi) 25° 23′ (vii) 41° 12′ (viii) 43° 24′ (ix) 28° 2′ (x) 42° 14′.

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1. 31·48. 2. ·6159. 3. $x \leqslant 2·026$. 4. 2·303 5. (i) $9·6786 - 10$ (ii) ·3868 (iii) ·2873 (iv) ·1086. 6. 14·2 years 7. $x = -11/3, -13/9$ 8. 11·58 hours 9. $-15·59$ 13. ·04844 14. ·6183 15. 19 digits 16. 19th from the left 17. ·9762 18. 1651. 20. (i) $x = 3·162, ·01$ (ii) $x = 10^{100}$ 21. 9 years 22. $-·009192$. 23. No. 25 1847 metres

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1. (i) $A = 60^\circ$, $B = 45^\circ$, $C = 75^\circ$.
(ii) $A = 120^\circ$, $B = 45^\circ$, $C = 15^\circ$.
(iii) $A = 49^\circ$, $B = 59^\circ$, $C = 72^\circ$.
(iv) $A = 20^\circ\ 55'$, $B = 25^\circ\ 50'$, $C = 132^\circ\ 35'$

2. $x > 1$.

3. (i) $A = 84^\circ$, $B = 34^\circ$, $C = 62^\circ$.
(ii) $A = 43^\circ$, $B = 61^\circ$, $C = 76^\circ$.
(iii) $A = 33^\circ$, $B = 53^\circ$, $C = 95^\circ$.

4. (i) $A = 56^\circ$, $B = 60^\circ$, $C = 64^\circ$.
(ii) $A = 60^\circ$, $B = 43^\circ\ 20'$, $C = 76^\circ\ 40'$.
(iii) $A = 60^\circ\ 9'$, $B = 43^\circ\ 17'$, $C = 76^\circ\ 34'$.
(iv) $A = 38^\circ\ 56'$, $B = 57^\circ\ 58'$, $C = 83^\circ\ 6'$.

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1. (i) $a = 1$, $B = 30^\circ$, $C = 120^\circ$.
(ii) $a = \sqrt{6}$, $B = 105^\circ$, $C = 15^\circ$.
(iii) $c = 13$, $A = 31^\circ$, $B = 112^\circ$.
(iv) $b = 15$, $A = 28^\circ$, $C = 117^\circ$.

3. (i) $a = 436$, $B = 78^\circ\ 15'$, $C = 49^\circ\ 35'$.
(ii) $c = 27{\cdot}29$, $A = 91^\circ\ 54'$, $B = 37^\circ\ 18'$.
(iii) $b = 199^01$, $A = 83^\circ\ 7'$, $C = 42^\circ\ 17'$.
(iv) $c = 226{\cdot}9$, $A = 39^\circ\ 45'$, $B = 73^\circ\ 35'$.

4. $A = 94^\circ\ 40'$, $B = 25^\circ\ 20'$.

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1. (i) $A = 106^\circ$, $b = 15$, $c = 120$.
(ii) $C = 47^\circ$, $a = 187$, $c = 139$.
(iii) $B = 97^\circ\ 50'$, $a = 169$, $b = 234$.
(iv) $C = 82^\circ\ 30'$, $b = 103$, $c = 172$.

3. (i) $C = 72^\circ$, $a = 85$, $c = 82$.
(ii) $A = 78^\circ\ 10'$, $a = 144$, $b = 83$.
(iii) $C = 65^\circ\ 45'$, $b = 22{\cdot}66$, $c = 21{\cdot}63$.
(iv) $A = 33^\circ\ 49'$, $a = 16{\cdot}18$, $c = 25{\cdot}63$.

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1. (i) $B = 35^\circ$, $C = 37^\circ$, $c = 18$.
(ii) $B = 22^\circ$, $C = 129^\circ$, $c = 51$.

(iii) $A_1 = 58^\circ$, $C_1 = 85^\circ$, $c_1 = 81$.
and $A_2 = 122^\circ$, $C_2 = 21^\circ$, $c_2 = 29$.
(iv) No solution.
(v) No solution.
(vi) $B_1 = 59^\circ$, $C_1 = 89^\circ$, $c_1 = 83$.
and $B_2 = 121^\circ$, $C_2 = 27^\circ$, $c_2 = 37$.
(vii) $B = 5^\circ\ 50'$, $C = 108^\circ\ 40'$, $c = 783$.
(viii) $B = 37^\circ\ 23'$, $C = 91^\circ\ 27'$, $c = 41{\cdot}34$.
(ix) $B_1 = 67^\circ\ 20'$, $C_1 = 60^\circ\ 10'$, $c_1 = 247{\cdot}1$.
and $B_2 = 112^\circ\ 40'$, $C_2 = 23^\circ\ 50'$, $c_2 = 106{\cdot}9$.
(x) $B = 50^\circ\ 10'$, $C = 28^\circ\ 30'$, $c = 11{\cdot}5$.
(xi) $B_1 = 66^\circ\ 40'$, $C_1 = 79^\circ\ 40'$, $c_1 = 581{\cdot}2$.
and $B_2 = 113^\circ\ 20'$, $C_2 = 33^\circ$, $c_1 = 321$.
(xii) No solution.
(xiii) No solution.
(xiv) $B = 43^\circ\ 51'$, $C = 83^\circ\ 57'$, $c = 36{\cdot}95$.

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1. 2·12 metres.
2. 14°.
3. 29 metres.
4. 87·1 metres.
5. 444 metres.
6. 430 metres.
7. height = 35 metres, distance = 20 metres.
8. 80 m.
9. height = 60 m, distance = 35 m.
10. The point is at a distance of 1205 m from one pole and the height of the pole is 21·7 metres.
11. 12 metres.
12. 15 minutes.
13. 33·3 metres.
14. 4·2 metres.
15. 65°.
16. 261 metres.

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1. 45 metres.
2. 7·53 km from P and 9·69 km from Q.
3. 43 km.
4. 54 metres.
5. 620 metres from A, 440 metres from B and 370 metres is the height of the balloon.
6. 8 m.
7. 8·8 m.
8. 62 m.
9. 11 metres.

10. 840 metres.

11. Elevation of the Sun = 37° and angle made by hillside with the horizontal = 2°.

12. 13°.

13. $l \sin (\alpha - \gamma) \sin (\beta - \alpha) \csc (\beta + \gamma) \sec \alpha$, if the points are on the same side of the tree; $l \sin (\alpha - \gamma) \sin (\beta - \alpha) \csc (\beta - \gamma) \sec \alpha$, if the points are on opposite sides of the tree, [Read 'a point l metres . . .' instead of 'a point l metre . . .'].

14. R can be either 290 km away or 140 km away from Q.

15. 60 metres.

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1. 20 metres. 2. 244 metres. 3. 35°.

4. the distance = 58 metres and the angle of elevation = 14°.

5. 330 metres.

6. The height of the metre or = 100 metres and its distance from A is 110 metres.

10. $\frac{a \sin \Upsilon \sin \beta}{\sin (\beta + \Upsilon)}$ feet.

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1. (i) $A = 40^\circ$, $B = 46^\circ$, $C = 94^\circ$.

(ii) $A = 37^\circ$, $B = 97^\circ$, $C = 46^\circ$.

(iii) $A = 116^\circ$, $B = 40^\circ$, $c = 32$.

(iv) $A = 79^\circ$, $b = 5{\cdot}5$, $c = 6{\cdot}1$.

(v) $B = 22^\circ$, $C = 107^\circ$, $c = 31$.

(vi) $B_1 = 78^\circ$, $C_1 = 40^\circ$, $c_1 = 33$.

$B_2 = 102^\circ$, $C_2 = 16^\circ$, $c_2 = 15$.

(vii) $C = 36^\circ$, $A = 44^\circ$, $a = 28$.

(viii) $A_1 = 46^\circ$, $B_1 = 105^\circ$, $b_1 = 45$.

$A_2 = 134^\circ$, $B_2 = 17^\circ$, $b_2 = 1{\cdot}5$.

2. (i) $A = 70^\circ\ 12'$, $B = 59^\circ\ 36'$, $C = 50^\circ\ 12'$.

(ii) $A = 84^\circ\ 50'$, $B = 54^\circ\ 50'$, $c = 366$.

(iii) $A = 70^\circ\ 50'$, $b = 65{\cdot}14$, $c = 77{\cdot}41$.

(iv) $C = 26^\circ\ 46'$, $A = 15^\circ\ 33'$, $a = 14{\cdot}70$.

(v) $A_1 = 59^\circ\ 34'$, $B_1 = 84^\circ\ 56'$, $b_1 = 66{\cdot}01$.

$A_2 = 120^\circ\ 26'$, $B_2 = 24^\circ\ 4'$, $b_2 = 27{\cdot}02$.

(vi) $C_1 = 61^\circ\ 24'$, $B_1 = 85^\circ\ 49'$, $b_1 = 604{\cdot}4$.

$C_2 = 118^\circ\ 36'$, $B_2 = 29^\circ\ 7'$, $b_2 = 292{\cdot}9$.

(vii) $C_1 = 24° 34'$, $A_1 = 134° 17'$, $a_1 = 84{\cdot}10$.
$C_1 = 155° 26'$, $A_2 = 3° 25'$, $a_2 = 7{\cdot}00$.

3. 30°.
4. 120°.
7. height of the hill above AB = 239 metres.
8. 13030 metres.
9. 3·8 metres.
11. 112·3 metres.
13. $h = \dfrac{d \sin \alpha \sin(\Upsilon + \beta)}{\sin(\Upsilon - \alpha)}$.
14. 144 metres.
15. Speed of the aeroplane = 490 km. per hour (correct to two significant figures).
Direction of Flight is S73°E, to the nearest degree.

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1. $-\pi/2, \pi/2, 3\pi/2, 5\pi/2$.
2. $0, \pi, 2\pi, 3\pi$.
3. $-3\pi/2, -\pi/2, \pi/2, 3\pi/2$.
4. $0, \pi, 2\pi, 3\pi$.
8. (i) $\{(x, \tan x) : 0 \leqslant x < \pi/2\}$; $\{(x, \sin x) : 0 \leqslant x < \pi/4\}$.
(ii) $\{(x, \cot x) : 0 < x \leqslant \pi/2\}$; $\{(x, \cos x) : 0 < x \leqslant \pi/4\}$.
(iii) $\{(x, x) : 0 \leqslant x < 1\}$; $\{(x, x^2) : 0 \leqslant x < 2\}$.
(iv) $\{(x, -x) : 0 \leqslant x < 1\}$; $\{(x, -x^2) : 0 \leqslant x < 2\}$.

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1. (i) $\sin^{-1}(-1) = -\pi/2$;
(ii) $\sin^{-1}(-\sqrt{3}/2 = -\pi/3$;
(iii) $\sin^{-1}(-1/\sqrt{2}) = \dfrac{-\pi}{4}$;
(iv) $\sin^{-1}(-1/2) = \dfrac{-\pi}{6}$;
(v) $\sin^{-1} 0 = 0$;
(vi) $\sin^{-1} \frac{1}{2} = \dfrac{\pi}{6}$;
(vii) $\sin^{-1}(1/\sqrt{2}) = \dfrac{\pi}{4}$;
(viii) $\sin^{-1}(\sqrt{3}/2) = \dfrac{\pi}{3}$;
(ix) $\sin^{-1} 1 = \dfrac{\pi}{2}$.

2. (i) $\cos^{-1}(-1) = \pi$;
(ii) $\cos^{-1}(-\sqrt{3}/2) = \dfrac{5\pi}{6}$;
(iii) $\cos^{-1}(-1/\sqrt{2}) = \dfrac{3\pi}{4}$;
(iv) $\cos^{-1}(-1/2) = \dfrac{2\pi}{3}$;
(v) $\cos^{-1} 0 = = \dfrac{\pi}{2}$;
(vi) $\cos^{-1} \frac{1}{2} = \dfrac{\pi}{3}$;
(vii) $\cos^{-1}(1/\sqrt{2}) = \dfrac{\pi}{4}$;
(viii) $\cos^{-1}(\sqrt{3}/2) = \dfrac{\pi}{6}$;
(ix) $\cos^{-1} 1 = 0$.

3. (i) $\tan^{-1}(-\sqrt{3}) = \frac{-\pi}{3}$;

(ii) $\tan^{-1}(-1) = \frac{-\pi}{4}$;

(iii) $\tan^{-1}(-1/\sqrt{3}) = \frac{-\pi}{6}$;

(iv) $\tan^{-1} 0=0$;

(v) $\tan^{-1}(-1/\sqrt{3})= \frac{-\pi}{6}$;

(vi) $\tan^{-1}(-1) = \frac{\pi}{4}$;

(vii) $\tan^{-1} \sqrt{3} = \frac{\pi}{3}$.

4. (i) $\cot^{-1}(-\sqrt{3}) = \frac{5\pi}{6}$;

(ii) $\cot^{-1}(-1) = \frac{3\pi}{4}$;

(iii) $\cot^{-1}(-1/\sqrt{3}) = \frac{2\pi}{3}$;

(iv) $\cot^{-1} 0= \frac{\pi}{2}$;

(v) $\cot^{-1}(1/\sqrt{3}) = \frac{\pi}{3}$;

(vi) $\cot^{-1} 1 = \frac{\pi}{4}$;

(vii) $\cot^{-1} \sqrt{3} = \frac{\pi}{6}$.

5. (i) $\sec^{-1}(-1) =\pi$;

(ii) $\sec^{-1}(-2/\sqrt{3})= \frac{5\pi}{6}$;

(iii) $\sec^{-1}(-\sqrt{2}) = \frac{3\pi}{4}$;

(iv) $\sec^{-1}(-2) = \frac{2\pi}{3}$;

(v) $\sec^{-1} 2= \frac{\pi}{3}$;

(vi) $\sec^{-1} \sqrt{2} = \frac{\pi}{4}$;

(vii) $\sec^{-1}(2/\sqrt{3}) = \frac{\pi}{6}$;

(viii) $\sec^{-1} 1=0$.

6. (i) $\csc^{-1}(-1) = -\frac{\pi}{2}$;

(ii) $\csc^{-1}\frac{-2}{\sqrt{3}} = \frac{-\pi}{3}$;

(iii) $\csc^{-1}(-\sqrt{2}) = \frac{-\pi}{4}$;

(iv) $\csc^{-1}(-2) = \frac{-\pi}{6}$;

(v) $\csc^{-1} 2 = \frac{\pi}{6}$;

(vi) $\csc^{-1} \sqrt{2} = \frac{\pi}{4}$;

(vii) $\csc^{-1}(2/\sqrt{3}) = \frac{\pi}{3}$;

(viii) $\csc^{-1} 1 = \frac{\pi}{2}$.

7. $\frac{12}{13}$. 8. $\frac{24}{25}$. 9. $5/\sqrt{11}$. 10. $\frac{4}{9}$.

11. $\frac{3}{5}$. 12. $-5\sqrt{74}$. 13. $-\frac{4\sqrt{6}}{5}$. 14. $-\frac{7}{24}$.

20. $\sin^{-1} \frac{4}{\sqrt{(x^2+16)}}$, $\tan^{-1} \frac{4}{x}$.

21. $\cos^{-1} \frac{3}{\sqrt{(x^2+9)}}$, $\tan^{-1} \frac{x}{3}$.

22. $\sin^{-1}\frac{x}{2}$, $\cos^{-1}\frac{\sqrt{(4-x^2)}}{2}$.

23. $\sin^{-1}\frac{3}{x}$, $\cos^{-1}\frac{\sqrt{(x^2-9)}}{x}$.

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1. (i) $\frac{63}{65}$; (ii) $\frac{4\sqrt{161}-24}{75}$; (iii) $\frac{1}{2}$;
(iv) $\frac{204}{325}$; (v) not defined; (vi) $\sqrt{3}$.

4. (i) $x \in \{a : a^2 \leqslant 1 \text{ and } a \neq o\}$;
(ii) $x \in \{a : a^2 \leqslant 1\}$;
(iii) $x \in \{a : a^2 \leqslant 1\}$;
(iv) $x \in \mathbf{R}$;
(v) $x \in \mathbf{R}$;
(vi) $x \in \{a : a^2 \geqslant 0, a \neq \pm 1\}$;
(vii) $x \in \{a : a^2 \leqslant 1\}$;
(viii) $x \in \{a : a^2 \leqslant 1\}$;
(ix) $x = \frac{1}{6}$;
(x) $x = \sqrt{3/28}$.

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1. (i) $\frac{4}{5}$; (ii) $-\frac{1}{2}$; (iii) $\frac{16}{65}$; (iv) $\frac{140}{171}$.

2. (i) $\left\{\frac{1+\sqrt{(1-4x^2)}}{2}\right\}^{\frac{1}{2}}$; (ii) $\frac{x^2}{1+x^2}$; (iii) x^2; (iv) $\left\{\frac{1+x}{2}\right\}^{\frac{1}{2}}$.

4. (i) $x = 1$; (ii) $x = \frac{1}{2}$; (iii) $x = \frac{1}{3}$; (iv) $x = \frac{1}{2}(2+\sqrt{2})^{\frac{1}{2}}$.

6. No, it is false.

7. No, it is meaningless.

8. No, it is does not make sense.

9. Yes; no.

10. $x \in \left\{a : o \leqslant a^2 < \frac{1}{3}\right\}$.

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1. $x \in \{ n\pi : n \in \mathbf{Z} \}$.

2. $x \in \{ (2n+1)\frac{\pi}{2} : n \in \mathbf{Z} \}$.

3. (i) $x \in \{ n\pi + (-1)^n \frac{\pi}{6} : n \in \mathbf{Z} \}$.

   (ii) $x \in \{ n\pi - (-1)^n \frac{\pi}{3} : n \in \mathbf{Z} \}$.

   (iii) $x \in \left\{ n\pi - (-1)^n \frac{\pi}{4} : n \in \mathbf{Z} \right\}$;

   (iv) $x \in \{ n\pi - (-1)^n \pi/6 : n \in \mathbf{Z} \}$;

   (v) $x \in \{ 2n\pi : n \in \mathbf{Z} \}$;

   (vi) $x \in \left\{ 2n\pi \pm 2\frac{\pi}{3} : n \in \mathbf{Z} \right\}$;

   (vii) $x \in \left\{ 2n\pi \pm 5\pi/6 : n \in \mathbf{Z} \right\}$;

   (viii) $x \in \{ 2n\pi \pm \pi/6 : n \in \mathbf{Z} \}$;

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1. (i) $\left\{ n\pi + (-1)^n \frac{\pi}{3} : n \in \mathbf{Z} \right\}$;

   (ii) $\left\{ 2n\pi \pm 2\frac{\pi}{3} : n \in \mathbf{Z} \right\}$.

   (iii) $\left\{ n\pi + \frac{\pi}{4} : n \in \mathbf{Z} \right\}$;

   (iv) $\left\{ n\pi - \frac{\pi}{6} : n \in \mathbf{Z} \right\}$.

   (v) $\left\{ 2n\pi \pm \frac{\pi}{3} : n \in \mathbf{Z} \right\}$;

   (vi) $\left\{ n\pi - (-1)^n \frac{\pi}{6} : n \in \mathbf{Z} \right\}$;

   (vii) $\left\{ 2n\pi \pm \cos^{-1}\frac{1}{4} : n \in \mathbf{Z} \right\}$;

   (viii) $\{ n\pi + \tan^{-1} 5 : n \in \mathbf{Z} \}$.

2. (i) $\left\{n\pi - (-1)^n \frac{\pi}{4} : n \in \mathbf{Z}\right\}$;

(ii) $\left\{2n\pi \pm \frac{\pi}{6} : n \in \mathbf{Z}\right\}$;

(iii) $\left\{\frac{n\pi}{2} + \frac{\pi}{12} : n \in \mathbf{Z}\right\}$;

(iv) $\left\{\frac{n\pi}{3} - \frac{1}{3}\tan^{-1}\frac{1}{2} : n \in \mathbf{Z}\right\}$;

(v) $\left\{2n\pi \pm \frac{\pi}{4} : n \in \mathbf{Z}\right\}$;

(vi) $\left\{n\pi - (-1)^n \sin^{-1} \frac{1}{6} : n \in \mathbf{Z}\right\}$.

3. $\left\{2n\pi \pm \frac{5\pi}{6} : n \in \mathbf{Z}\right\}$.

4. $\left\{n\pi - \frac{\pi}{4} : n \in \mathbf{Z}\right\}$.

5. (i) $x \in \left\{2n\pi \pm \frac{\pi}{3} : n \in \mathbf{Z}\right\}$;

(ii) $x \in \left\{n\pi - (-1)^n \frac{\pi}{4} : n \in \mathbf{Z}\right\}$;

(iii) $x \in \left\{\frac{n\pi}{2} - \frac{\pi}{12} : n \in \mathbf{Z}\right\}$;

(iv) $x \in \left\{\frac{n\pi}{4} - \frac{1}{4}\tan^{-1}(5/2) : n \in \mathbf{Z}\right\}$.

12. A. 13. A.

14. (i) $\left\{\frac{-5\pi}{3}, \frac{-\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}\right\}$;

(ii) $\left\{\frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}\right\}$;

(iii) $\left\{\frac{-8\pi}{3}, \frac{-7\pi}{3}, \frac{-2\pi}{3}, \frac{-\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}\right\}$;

(iv) $\left\{\frac{-10\pi}{3}, \frac{-8\pi}{3}, \frac{-4\pi}{3}, \frac{-2\pi}{3}\right\}$.

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1. $x \in \left\{n\pi \pm \frac{\pi}{3} : n \in \mathbf{Z}\right\}$.
2. $x \in \left\{n\pi \pm \frac{\pi}{3} : n \in \mathbf{Z}\right\}$.
3. $x \in \left\{n\pi \pm \frac{\pi}{6} : n \in \mathbf{Z}\right\}$.
4. $x \in \left\{n\pi \pm \frac{\pi}{6} : n \in \mathbf{Z}\right\}$.
5. $x \in \left\{n\pi \pm \frac{\pi}{3} : n \in \mathbf{Z}\right\}$.
6. $x \in \left\{n\pi \pm \frac{\pi}{4} : n \in \mathbf{Z}\right\}$.
7. $x \in \left\{n\pi - (-1)^n \frac{\pi}{6} : n \in \mathbf{Z}\right\}$.
8. $x \in \left[\left\{n\pi + (-1)^n \frac{\pi}{2} : n \in \mathbf{Z}\right\} \cup \left\{n\pi + (-1)^n \frac{\pi}{6} : n \in \mathbf{Z}\right\}\right]$.
9. $x \in \left[\left\{(2n+1)\pi : n \in \mathbf{Z}\right\} \cup \left\{2n\pi \pm \frac{2\pi}{3} : n \in \mathbf{Z}\right\}\right]$.
10. $x \in \left[\left\{n\pi + \frac{\pi}{4} : n \in \mathbf{Z}\right\} \cup \left\{n\pi - \tan^{-1} 2 : n \in \mathbf{Z}\right\}\right]$.
11. $x \in \left[\left\{n\pi + \frac{\pi}{4} : n \in \mathbf{Z}\right\} \cup \left\{n\pi - \tan^{-1} \frac{1}{3} : n \in \mathbf{Z}\right\}\right]$.
12. $x \in \left[\left\{n\pi - (-1)^n \frac{\pi}{6} : n \in \mathbf{Z}\right\} \cup \left\{n\pi + (-1)^n \sin^{-1} \frac{1}{3} : n \in \mathbf{Z}\right\}\right]$.
13. $x \in [\{2n\pi : n \in \mathbf{Z}\} \cup \{2n\pi \pm \cos^{-1}(-\frac{1}{4}) : n \in \mathbf{Z}\}]$.
14. $x \in \left\{n\pi + (-1)^n \frac{\pi}{6} : n \in \mathbf{Z}\right\}$.
15. $x \in \left\{2n\pi \pm \frac{\pi}{3} : n \in \mathbf{Z}\right\}$.
16. $x \in \left\{2n\pi \pm \frac{\pi}{3} : n \in \mathbf{Z}\right\}$.
17. $x \in \left\{2n\pi \pm \frac{\pi}{3} : n \in \mathbf{Z}\right\}$.

18. $x \in \left[\left\{(2n+1)\frac{\pi}{2} : n \in \mathbf{Z}\right\} \cup \left\{n\pi \pm \frac{\pi}{3} : n \in \mathbf{Z}\right\}\right].$

19. $x \in \left[\left\{\frac{n\pi}{2} \pm \frac{\pi}{12} : n \in \mathbf{Z}\right\}\right].$

20. $x \in \left[\left\{\frac{(2n+1)\pi}{2} : n \in \mathbf{Z}\right\} \cup \left\{2n\pi : n \in \mathbf{Z}\right\}\right].$

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1. $x \in \left\{\frac{n\pi}{5} : n \in \mathbf{Z}\right\}.$

2. $x \in \left[\left\{n\pi : n \in \mathbf{Z}\right\} \cup \left\{(2n+1)\frac{\pi}{8} : n \in \mathbf{Z}\right\}\right].$

3. $x \in \left\{(2n+1)\frac{\pi}{18} : n \in \mathbf{Z}\right\}.$

4. $x \in \left[\left\{\frac{n\pi}{2} : n \in \mathbf{Z}\right\} \cup \left\{\frac{(2n+1)\pi}{8} : n \in \mathbf{Z}\right\}\right].$ [Read sin 6 x instead of 6 x.]

5. $x \in \{n\pi : n \in \mathbf{Z}\}.$

6. $x \in \left\{(2n+1)\frac{\pi}{12} : n \in \mathbf{Z}\right\}.$

7. $x \in \left[\left\{(4n-1)\frac{\pi}{2} : n \in \mathbf{Z}\right\} \cup \left\{\frac{(4n+3)\pi}{10} : n \in \mathbf{Z}\right\}\right].$

8. $x \in \left\{\frac{(2n+1)\pi}{p \pm q} : n \in \mathbf{Z}\right\}.$

9. $x \in \left[\left\{\frac{2n\pi}{p+q} : n \in \mathbf{Z}\right\} \cup \left\{\frac{(2n+1)\pi}{p-q} : n \in \mathbf{Z}\right\}\right].$

10. $x \in \left\{\frac{2k+1}{m+n}\frac{\pi}{2} : k \in \mathbf{Z}\right\}.$

11. $x \in \left[\left\{\frac{(2n+1)\pi}{4} : n \in \mathbf{Z}\right\} \cup \left\{2n\pi \pm \pi/3 : n \in \mathbf{Z}\right\}\right].$

12. $x \in \left[\left\{\frac{(2n+1)\pi}{8} : n \in \mathbf{Z}\right\} \cup \left\{\frac{2n\pi}{3} \pm \pi/9 : n \in \mathbf{Z}\right\}\right].$

13. $x \in \left[\left\{\frac{n\pi}{3} : n \in \mathbf{Z}\right\} \cup \left\{\frac{n\pi}{2} \pm \frac{\pi}{12} : n \in \mathbf{Z}\right\}\right].$

14. $x \in \left[\left\{\frac{(2n+1)\pi}{6} : n \in \mathbf{Z}\right\} \cup \left\{n\pi \pm \frac{\pi}{3} : n \in \mathbf{Z}\right\}\right].$

15. $x \in \left[\left\{\frac{(2n+1)\pi}{4} : n \in \mathbf{Z}\right\} \cup \left\{2n\pi \pm \frac{2\pi}{3} : n \in \mathbf{Z}\right\}\right].$

16. $x \in \left[\left\{\frac{n\pi}{3} : n \in \mathbf{Z}\right\} \cup \left\{\frac{n\pi}{2} - (-1)^n \frac{\pi}{12} : n \in \mathbf{Z}\right\}\right].$

17. $x \in \left[\left\{\frac{2n\pi}{5} : n \in \mathbf{Z}\right\} \cup \left\{\frac{(2n+1)\pi}{2} : n \in \mathbf{Z}\right\} \cup \left\{(2n+1)\pi : n \in \mathbf{Z}\right\}\right].$

18. $x \in \left\{\frac{n\pi}{4} : n = -8, -7, \ldots, 0, 1, 2, \ldots, 6, 7, 8\right\}.$

19. $x \in \left\{\frac{n\pi}{6} : n = -5, -4, \ldots, -3, -2, -1, 1, 2, 3, 4, 5\right\}.$

20. $x \in \left\{\frac{n\pi}{3} : n = 0, 1, 2, 3, 4, 5, 6\right\}.$

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1. (i) $x \in \left\{2n\pi + \frac{2\pi}{3} : n \in \mathbf{Z}\right\}$;

   (ii) $x \in \left\{2n\pi + \frac{5\pi}{6} : n \in \mathbf{Z}\right\}$;

   (iii) $x \in \left\{2n\pi - \frac{\pi}{4} : n \in \mathbf{Z}\right\}$;

   (iv) $x \in \phi$;

   (v) $x \in \left\{2n\pi + \frac{2\pi}{3} : n \in \mathbf{Z}\right\}$;

   (vi) $x \in \left\{2n\pi - \frac{\pi}{3} : n \in \mathbf{Z}\right\}$;

   (vii) $x \in \left\{2n\pi - \frac{\pi}{3} : n \in \mathbf{Z}\right\}$.

2. $x \in \left\{-\frac{2\pi}{3}, \frac{4\pi}{3}\right\}$.

3. (i) $\{(x, y) : x = (m\pi \pm \pi/8) + (n\pi \pm \pi/6),\ y = (m\pi \pm \pi/8) - (n\pi \pm \pi/6),\ m \in \mathbf{Z},\ n \in \mathbf{Z}\}$;

   (ii) $\left\{(x, y) : x = \frac{2}{3}(2m\pi \pm \pi/3) - \frac{1}{3}(2n\pi \pm \pi/6),\right.$

   $\left. y = -\frac{1}{3}(2m\pi \pm \pi/3) + \frac{2}{3}(2n\pi \pm \pi/6),\ m, n \in \mathbf{Z}\right\}$;

(iii) $\left\{ (x, y) : x = \frac{1}{3} (m\pi + (-1)^m \pi/4) + \frac{1}{3} (2n\pi \pm 5\pi/6), \right.$

$\left. y = \frac{2}{3} (m\pi + (-1)^m \pi/4) - \frac{1}{3} (2n\pi \pm 5\pi/6), m \in \mathbf{Z}, n \in \mathbf{Z} \right\};$

(iv) $\left\{ (x, y) : x = \frac{1}{10} (2m\pi \pm \pi/3) + \frac{3}{10} (n\pi + \pi/6), \right.$

$\left. y = \frac{3}{10} (2m\pi \pm \pi/3) - \frac{1}{10} (n\pi + \pi/6), m \in \mathbf{Z}, n \in Z \right\};$

(v) $\{(\pi/24, 5\pi/24), (5\pi/24, \pi/24), (7\pi/24, 11\pi/24), (11\pi/24, 7\pi/24)\}$.

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1. (i) $\{x : -\pi/2 \leqslant x \leqslant \pi/6\} \cup \{x : 5\pi/6 \leqslant x \leqslant 3\pi/2\}$;
   (ii) $\{x : -\pi/3 < x < \pi/3\} \cup \{x : 2\pi/3 < x < 4\pi/3\}$;
   (iii) $\{x : -\pi/2 \leqslant x < 0\} \cup \{x : \pi/4 < x \leqslant 3\pi/4\} \cup \{x : \pi < x \leqslant 3\pi/2\}$;
   (iv) $\{x : -\pi/2 \leqslant x < -\pi/4\} \cup \{z : \pi/4 < x < 3\pi/4\} \cup \{x : 5\pi/4 < x < 3\pi/2\}$.

2. (i) $\{x : \pi/3 < x < 5\pi/3\}$;
   (ii) $\{x : 0 < x < \pi/3\} \cup \{x : 2\pi/3 < x < 4\pi/3\} \cup \{x : 5\pi\ 3 < x < 2\pi\}$;
   (iii) $\{x : 0 \leqslant x \leqslant \pi/12\} \cup \{x : 7\pi/12 \leqslant x \leqslant 2\pi\}$;
   (iv) $\{x : 4\pi/3 \leqslant x \leqslant 2\pi\}$.

3. (i) $\{x : -\pi/4 \leqslant x \leqslant \pi/4\}$;
   (ii) $\{x : -\pi/6 < x < \pi/6\}$;
   (iii) $\{x : -\pi/2 \leqslant x < -\alpha\} \cup \{x : \alpha < x < \pi/2\}$, where $\alpha = \cot^{-1} 4$,
   (iv) $\{x : -\pi/2 < x \leqslant -\alpha\} \cup \{x : \alpha \leqslant x < \pi/2\}$, where $\alpha = \sec^{-1} \sqrt{10}$.

4. (i) $\{2n\pi + \alpha : n \in \mathbf{Z}$ and $\pi/4 < \alpha < 3\pi/4\}$;
   (ii) $\{n\pi + \alpha : n \in \mathbf{Z}$ and $\pi/4 \leqslant \alpha \leqslant 3\pi/4\}$;
   (iii) $\{n\pi + \alpha : n \in \mathbf{Z}$ and $\pi/3 \leqslant \alpha \leqslant 2\pi/3\}$;
   (iv) $\{n\pi + \alpha : n \in \mathbf{Z}$ and either $\pi/3 < \alpha < \pi/2$ or $\pi/2 < \alpha < 2\pi/3)\}$;
   (v) $\{n\pi + \alpha : n \in \mathbf{Z}$ and $-\tan^{-1}(1/3) \leqslant \alpha \leqslant \tan^{-1}(1/3)\}$;
   (vi) $\{2n\pi + \pi/4 + \alpha : n \in \mathbf{Z}$ and $\pi/4 \leqslant \alpha \leqslant 7\pi/4\}$;
   (vii) $\{2n\pi + \cos^{-1}(4/5) + \theta : n \in \mathbf{Z}$ and $\cos^{-1}(2/5) \leqslant \theta \leqslant 2\pi - \cos^{-1}(4/5)\}$;
   (viii) $\{n\pi + (-1)^n \theta : n \in \mathbf{Z}$ and $\sin^{-1}(1/3) \leqslant \theta \leqslant \pi/6\}$.

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1. (i) $x \in \{(2n+1)\pi/4 : n \in \mathbf{Z}\}$;
   (ii) $x \in \{n\pi + \pi/4 \pm \pi/6 : n \in \mathbf{Z}\}$;
   (iii) $x \in [\{n\pi/2 : n \in \mathbf{Z}\} \cup \{n\pi/3 : n \in \mathbf{Z}\}]$;

(iv) $x \in [\{ n\pi = n \in \mathbf{Z} \} \cup \{ 2n\pi \pm \cos^{-1}(1/4) : n \in \mathbf{Z} \}]$;

(v) $x \in \{ 2n\pi \pm \pi/3 : n \in \mathbf{Z} \}$;

(vi) $n \in \{ n\pi/2 + \pi/8 : n \in \mathbf{Z} \}$;

(vii) $x \in \{ n\pi \pm \alpha : n \in \mathbf{Z} \}$, where $\alpha = \tan^{-1}\left\{\dfrac{1 + \sqrt{21}}{10}\right\}$;

(viii) $x \in [\{ 2n\pi : n \in \mathbf{Z} \} \cup \{(4n + 1)\pi/6 : n \in \mathbf{Z} \}]$;

(ix) $x \in [\{ 2n + 1)\pi : n \in \mathbf{Z} \} \cup \{(2n\pi \pm \pi/3) : n \in \mathbf{Z} \}]$;

(x) $x \in [\{ 2n\pi : n \in \mathbf{Z} \} \cup \{ 2n\pi - 2\cos^{-1}\left(\dfrac{1}{\sqrt{5}}\right) : n \in \mathbf{Z} \}]$;

(xi) $x \in \{(2n - 1)\pi/4 : n \in \mathbf{Z} \}$;

(xii) $x \in \phi$;

(xiii) $x \in [\{ n\pi + \pi/4 : n \in \mathbf{Z} \} \cup \{ n\pi + \pi/3 : n \in \mathbf{Z} \}]$;

(xiv) $x \in [\{ 2n\pi/5 : n \in \mathbf{Z} \} \cup \{ (2n + 1)\pi/2 : n \in \mathbf{Z} \} \cup \{ (2n + 1)\pi : n \in \mathbf{Z}\}$.

2. $x \in \{ 2n\pi + 2\pi/3 : n \in \mathbf{Z} \}$.

3. $\{ (x, y) : x = \frac{1}{8}(2m\pi \pm \pi/6) + \frac{1}{8}(n\pi + (-1)^n\pi/4), y = \frac{1}{8}(n\pi + (-1)^n \pi/4) - \frac{1}{8}(2m\pi \pm \pi/6), m \in \mathbf{Z}, n \in \mathbf{Z}\}$.

4. $\{ (11\pi/36, 13\pi/18), (5\pi/36, 19\pi/18), (53\pi/36, 7\pi/18), (35\pi/36, 25\pi/18), (29\pi/36, 31\pi/18), (43\pi/36, 5\pi/18), (37\pi/36, 11\pi/18)\}$.

5. $[\{ x : -\pi/2 \leqslant x \leqslant \pi/3 \} \cup \{ x : 2\pi/3 \leqslant x \leqslant 3\pi/2 \}]$.

6. $\{ n\pi \pm \alpha : 0 \leqslant \alpha < \pi/6\}$.

7. $\{ n\pi + \frac{1}{2}\cos^{-1}\left(\dfrac{1}{\sqrt{5}}\right) + \alpha : n \in \mathbf{Z}$ and $\pi/6 < \alpha < 5\pi/6 \}$.

TABLE 1

VALUES OF TRIGONOMETRIC FUNCTIONS

Angle	sin	cos	tan	cot	sec	csc	Angle
0°00′	·0000	1·0000	·0000	—	1·000	—	90°00′
10′	·0029	1·0000	·0029	343·8	1·000	343·8	50′
20′	·0058	1·0000	·0058	171·9	1·000	171·9	40′
30′	·0087	1·0000	·0087	114·6	1·000	114·6	30′
40′	·0116	·9999	·0116	85·94	1·000	85·95	20′
50′	·0145	·9999	·0145	68·75	1·000	68·76	10′
1°00′	·0175	·9998	·0175	57·29	1·000	57·30	89°00′
10′	·0204	·9998	·0204	49·10	1·000	49·11	50′
20′	·0233	·9997	·0233	42·96	1·000	42·98	40′
30′	·0262	·9997	·0262	38·19	1·000	38·20	30′
40′	·0291	·9996	·0291	34·37	1·000	34·38	20′
50′	·0320	·9995	·0320	31·24	1·001	31·26	10′
2°00′	·0349	·9994	·0349	28·64	1·001	28·65	88°00′
10′	·0378	·9993	·0378	26·43	1·001	26·45	50′
20′	·0407	·9992	·0407	24·54	1·001	24·56	40′
30′	·0436	·9990	·0437	22·90	1·001	22·93	30′
40′	·0465	·9989	·0466	21·47	1·001	21·49	20′
50′	·0494	·9989	·0495	20·21	1·001	20·23	10′
3°00′	·0523	·9986	·0524	19·08	1·001	19·11	87°00′
10′	·0552	·9985	·0553	18·07	1·002	18·10	50′
20′	·0581	·9983	·0582	17·17	1·002	17·20	40′
30′	·0610	·9981	·0612	16·35	1·002	16·38	30′
40′	·0640	·9980	·0641	15·60	1·002	15·64	20′
50′	·0669	·9978	·0670	14·92	1·002	14·96	10′
4°00′	·0698	·9976	·0699	14·30	1·002	14·34	86°00′
10′	·0727	·9974	·0729	13·73	1·003	13·76	50′
20′	·0756	·9971	·0758	13·20	1·003	13·23	40′
30′	·0785	·9969	·0787	12·71	1·003	12·75	30′
40′	·0814	·9967	·0816	12·25	1·003	12·29	20′
50′	·0843	·9964	·0846	11·83	1·004	11·87	10′
5°00′	·0872	·9962	·0875	11·43	1·004	11·47	85°00′
10′	·0901	·9959	·0904	11·06	1·004	11·10	50′
20′	·0929	·9957	·0934	10·71	1·004	10·76	40′
30′	·0958	·9954	·0963	10·39	1·005	10·43	30′
40′	·0987	·9951	·0992	10·08	1·005	10·13	20′
50′	·1016	·9948	·1022	9·788	1.005	9·839	10′
6°00′	·1045	·9945	·1051	9·514	1·006	9·567	84°00′
10′	·1074	·9942	·1080	9·225	1·006	9·309	50′
20′	·1103	·9939	·1110	9·010	1·006	9·065	40′
30′	·1132	·9936	·1139	8·777	1·006	8·834	30′
40′	·1161	·9932	·1169	8·556	1·007	8·614	20′
50′	·1190	·9929	·1198	8·345	1·007	8·405	10′
7°00′	·1219	·9925	·1228	8·144	1·008	8·206	83°00′
10′	·1248	·9922	·1257	7·953	1·008	8·016	50′
20′	·1276	·9918	·1287	7·770	1·008	7·834	40′
30′	·1305	·9914	·1317	7·596	1·009	7·661	30′
40′	·1334	·9911	·1346	7·429	1·009	7·496	20′
50′	·1363	·9907	·1376	7·269	1·009	7·337	10′
Angle	cos	sin	cot	tan	csc	sec	Angle

VALUES OF TRIGONOMETRIC FUNCTIONS *(Continued)*

Angle	sin	cos	tan	cot	sec	csc	Angle
8°00′	·1392	·9903	·1405	7·115	1·010	7·185	82°00′
10′	·1421	·9899	·1435	6·968	1·010	7·040	50′
20′	·1449	·9894	·1465	6·827	1·011	6·900	40′
30′	·1478	·9890	·1495	6·691	1·011	6·765	30′
40′	·1507	·9886	·1524	6·561	1·012	6·636	20′
50′	·1536	·9881	·1554	6·435	1·012	6·512	10′
9°00′	·1564	·9877	·1584	6·314	1·012	6·392	81°00′
10′	·1593	·9872	·1614	6·197	1·013	6·277	50′
20′	·1622	·9868	·1644	6·084	1·013	6·166	40′
30′	·1650	·9863	·1673	5·976	1·014	6·059	30′
40′	·1679	·9858	·1703	5·871	1·014	5·955	20′
50′	·1708	·9853	·1733	5·769	1·015	5·855	10′
10°00′	·1736	·9848	·1763	5·671	1·015	5·759	80°00′
10′	·1765	·9843	·1793	5·576	1·016	5·665	50′
20′	·1794	·9838	·1823	5·485	1·016	5·575	40′
30′	·1822	·9833	·1853	5·396	1·017	5·487	30′
40′	·1851	·9827	·1883	5·309	1·018	5·403	20′
50′	·1880	·9822	·1914	5·226	1·018	5·320	10′
11°00′	·1908	·9816	·1944	5·145	1·019	5·241	79°00′
10′	·1937	·9811	·1974	5·066	1·019	5·164	50′
20′	·1965	·9805	·2004	4·989	1·020	5·089	40′
30′	·1994	·9799	·2035	4·915	1·020	5·016	30′
40′	·2022	·9793	·2065	4·843	1·021	4·945	20′
50′	·2051	·9787	·2095	4·773	1·022	4·876	10′
12°00′	·2079	·9781	·2126	4·705	1·022	4·810	78°00′
10′	·2108	·9775	·2156	4·638	1·023	4·745	50′
20′	·2136	·9769	·2186	4·574	1·024	4·682	40′
30′	·2164	·9763	·2217	4·511	1·024	4·620	30′
40′	·2193	·9757	·2247	4·449	1·025	4·560	20′
50′	·2221	·9750	·2278	4·390	1·026	4·502	10′
13°00′	·2250	·9744	·2309	4·331	1·026	4·445	77°00′
10′	·2278	·9737	·2339	4·275	1·027	4·390	50′
20′	·2306	·9730	·2370	4·219	1·028	4·336	40′
30′	·2334	·9724	·2401	4·165	1·028	4·284	30′
40′	·2363	·9717	·2432	4·113	1·029	4·232	20′
50′	·2391	·9710	·2462	4·061	1·030	4·182	10′
14°00′	·2419	·9703	·2493	4·011	1·035	4·134	76°00′
10′	·2447	·9696	·2524	3·962	1·031	4·086	50′
20′	·2476	·9689	·2555	3·914	1·032	4·039	40′
30′	·2504	·9681	·2586	3·867	1·033	3·994	30′
40′	·2532	·9674	·2617	3·821	1·034	3·950	20′
50′	·2560	·9667	·2648	3·776	1·034	3·906	10′
15°00′	·2588	·9659	·2679	3·732	1·035	3·864	75°00′
10′	·2616	·9652	·2711	3·689	1·036	3·822	50′
20′	·2644	·9644	·2742	3·647	1·037	3·782	40′
30′	·2672	·9636	·2773	3·606	1·038	3·742	30′
40′	·2700	·9628	·2805	3·566	1·039	3·703	20′
50′	·2728	·9621	·2836	3·526	1·039	3·665	10′
Angle	cos	sin	cot	tan	csc	sec	Angle

VALUES OF TRIGONOMETRIC FUNCTIONS *(Continued)*

Angle	sin	cos	tan	cot	sec	csc	Angle
16°00′	·2756	·9613	·2867	3·487	1·040	3·628	74°00′
10′	·2784	·9605	·2899	3·450	1·041	3·592	50′
20′	·2812	·9596	·2931	3·412	1·041	3·556	40′
30′	·2840	·9588	·2962	3·376	1·043	3·521	30′
40′	·2868	·9580	·2994	3.340	1·044	3·487	20′
50′	·2896	·9572	·3026	3·305	1·045	3·453	10′
17°00′	·2924	·9563	·3057	3·271	1·046	3·420	73°00′
10′	·2952	·9555	·3089	3·237	1·047	3·388	50′
20′	·2979	·9546	·3121	3·204	1·048	3·356	40′
30′	·3007	·9537	.3153	3·172	1·049	3·326	30′
40′	·3035	·9528	·3185	3·140	1·049	3.295	20′
50′	·3062	·9520	·3217	3·108	1·050	3·265	10′
18°00′	·3090	·9511	·3249	3.078	1·051	3·236	72°00′
10′	·3118	·9502	·3281	3·047	1·052	3·207	50′
20′	·3145	·9492	·3314	3·018	1·053	3·179	40′
30′	·3173	·9483	·3346	2·989	1·054	3·152	30′
40′	·3201	·9474	·3378	2·960	1·056	3·124	20′
50′	·3228	·9465	·3411	2·932	1·057	3.098	10′
19°00′	·3256	·9455	·3443	2·904	1·058	3·072	71°00′
10′	·3283	·9446	·3476	2·877	1·059	3·046	50′
20′	·3311	·9436	·3508	2·850	1·060	3·021	40′
30′	·3338	·9426	·3541	2·824	1·061	2·996	30′
40′	·3365	·9417	·3574	2·798	1·062	2·971	20
50′	·3393	·9407	·3607	2·773	1·063	2·947	10′
20°00′	·3420	·9397	·3640	2·747	1·064	2·924	70°00′
10′	·3448	·9387	·3673	2·723	1·065	2·901	50′
20′	·3475	·9377	·3706	2·699	1·066	2·878	40′
30′	·3502	·9367	·3739	2·675	1·068	2·855	30′
40′	·3529	·9356	·3772	2·651	1·069	2·833	20′
50′	·3557	·9346	·3805	2·628	1·070	2·812	10′
21°00′	·3584	·9336	·3839	2·605	1·071	2·790	69°00′
10′	·3611	·9325	·3872	2·583	1·072	2·769	50′
20′	·3638	·9315	·3906	2·560	1·074	2·749	40′
30′	·3665	·9304	·3939	2·539	1·075	2·729	30′
40′	·3692	·9293	·3973	2·517	1·076	2·709	20′
50′	·3719	·9283	·4006	2·496	1·077	2·689	10′
22°00′	·3746	·9272	·4040	2·475	1·079	2·669	68°00′
10′	·3773	·9261	·4074	2·455	1·080	2·650	50′
20′	·3800	·9250	·4108	2·434	1·081	2·632	40′
30′	·3827	·9239	·4142	2·414	1·082	2·613	30′
40′	·3854	·9228	·4176	2·394	1·084	2·595	20′
50′	·3881	·9216	·4210	2·375	1·085	2·577	10′
23°00′	·3907	·9205	·4245	2·356	1·086	2·559	67°00′
10′	·3934	·9194	·4279	2·337	1·088	2·542	50′
20′	·3961	·9182	·4314	2·318	1·089	2·525	40′
30′	·3987	·9171	·4348	2·300	1·090	2·508	30′
40′	·4014	·9159	·4383	2·282	1·092	2·491	20′
50′	·4041	·9147	·4417	2·264	1·093	2·475	10′
Angle	cos	sin	cot	tan	csc	sec	Angle

VALUES OF TRIGONOMETRIC FUNCTIONS *(Continued)*

Angle	sin	cos	tan	cot	sec	csc	Angle
24°00′	·4067	·9135	·4452	2·246	1·095	2·459	66°00′
10′	·4094	·9124	·4487	2·229	1·096	2·443	50′
20′	·4120	·9112	·4522	2·211	1·097	2·427	40′
30′	·4147	·9100	·4557	2·194	1·099	2·411	30′
40′	·4173	·9088	·4592	2·177	1·100	2·396	20′
50′	·4200	·9075	·4628	2·161	1·102	2·381	10′
25°00′	·4226	·9063	·4663	2·145	1·103	2·366	65°00′
10′	·4253	·9051	·4699	2·128	1·105	2·352	50′
20′	·4279	·9038	·4734	2·112	1·106	2·337	40′
30′	·4305	·9026	·4770	2·097	1·108	2·323	30′
40′	·4331	·9013	·4806	2·081	1·109	2·309	20′
50′	·4358	·9001	·4841	2·066	1·111	2·295	10′
26°00′	·4384	·8988	·4877	2·050	1·113	2·281	64°00′
10′	·4410	·8975	·4913	2·035	1·114	2·268	50′
20′	·4436	·8962	·4950	2·020	1·116	2·254	40′
30′	·4462	·8949	·4986	2·006	1·117	2·241	30′
40′	·4488	·8936	·5022	1·991	1·119	2·228	20′
50′	·4514	·8923	·5059	1·977	1·121	2·215	10′
27°00′	·4540	·8910	·5095	1·963	1·122	2·203	63°00′
10′	·4566	·8897	·5132	1·949	1·124	2·190	50′
20′	·4592	·8884	·5169	1·935	1·126	2·178	40′
30′	·4617	·8870	·5206	1·921	1·127	2·166	30′
40′	·4643	·8857	·5243	1·907	1·129	2·154	20′
50′	·4669	·8843	·5280	1·894	1·131	2·142	10′
28°00′	·4695	·8829	·5317	1·881	1·133	2·130	62°00′
10′	·4720	·8816	·5354	1·868	1·134	2·118	50′
20′	·4746	·8802	·5392	1·855	1·136	2·107	40′
30′	·4772	·8788	·5430	1·842	1·138	2·096	30′
40′	·4797	·8774	·5467	1·829	1·140	2·085	20′
50′	·4823	·8760	·5505	1·816	1·142	2·074	10′
29°00′	·4848	·8746	·5543	1·804	1·143	2·063	61°00′
10′	·4874	·8732	·5581	1·792	1·145	2·052	50′
20′	·4899	·8718	·5619	1·780	1·147	2·041	40′
30′	·4924	·8704	·5658	1·767	1·149	2·031	30′
40′	·4950	·8689	·5696	1·756	1·151	2·020	20′
50′	·4975	·8675	·5735	1·744	1·153	2·010	10′
30°00′	·5000	·8660	·5774	1·732	1·155	2·000	60°00′
10′	·5025	·8646	·5812	1·720	1·157	1·990	50′
20′	·5050	·8631	·5851	1·709	1·159	1·980	40′
30′	·5075	·8616	·5890	1·698	1·161	1·970	30′
40′	·5100	·8601	·5930	1·686	1·163	1·961	20′
50′	·5125	·8587	·5969	1·675	1·165	1·851	10′
31°00′	·5150	·8572	·6009	1·664	1·167	1·942	59°00′
10′	·5175	·8557	·6048	1·653	1·169	1·932	50′
20′	·5200	·8542	·6088	1·643	1·171	1·923	40′
30′	·5225	·8526	·6128	1·632	1·173	1·914	30′
40′	·5250	·8511	·6168	1·621	1·175	1·905	20′
50′	·5275	·8496	·6208	1·611	1·177	1·896	10′
Angle	cos	sin	cot	tan	csc	sec	Angle

Values of Trigonometric Functions *(Continued)*

Angle	sin	cos	tan	cot	sec	csc	Angle
32°00′	·5299	·8480	·6249	1·600	1·179	1·887	58°00′
10′	·5324	·8465	·6289	1·590	1·181	1·878	50′
20′	·5348	·8450	·6330	1·580	1·184	1·870	40′
30′	·5373	·8434	·6371	1·570	1·186	1·861	30′
40′	·5398	·8418	·6412	1·560	1·188	1·853	20′
50′	·5422	·8403	·6453	1·550	1·190	1·844	10′
33°00′	·5446	·8387	·6494	1·540	1·192	1·836	57°00′
10′	·5471	·8371	·6536	1·530	1·195	1·828	50′
20′	·5495	·8355	·6577	1·520	1·197	1·820	40′
30′	·5519	·8339	·6619	1·511	1·199	1·812	30′
40′	·5544	·8323	·6661	1·501	1·202	1·804	20′
50′	·5568	·8307	·6703	1·492	1·204	1·796	10′
34°00′	·5592	·8290	·6745	1·483	1·206	1·788	56°00′
10′	·5616	·8274	·6787	1·473	1·209	1·781	50′
20′	·5640	·8258	·6830	1·464	1·211	1·773	40′
30′	·5664	·8241	·6873	1·455	1·213	1·766	30′
40′	·5688	·8225	·6916	1·446	1·216	1·758	20′
50′	·5712	·8208	·6959	1·437	1·218	1·751	10′
35°00′	·5736	·8192	·7002	1·428	1·221	1·743	55°00′
10′	·5760	·8175	·7046	1·419	1·223	1·736	50′
20′	·5783	·8158	·7089	1·411	1·226	1·729	40′
30′	·5807	·8141	·7133	1·402	1·228	1·722	30′
40′	·5831	·8124	·7177	1·393	1·231	1·715	20′
50′	·5854	·8107	·7221	1·385	1·233	1·708	10′
36°00′	·5878	·8090	·7265	1·376	1·236	1·701	54°00′
10′	·5901	·8073	·7310	1·368	1·239	1·695	50′
20′	·5925	·8056	·7355	1·360	1·241	1·688	40′
30′	·5948	·8039	·7400	1·351	1·244	1·681	30′
40′	·5972	·8021	·7445	1·343	1·247	1·675	20′
50′	·5995	·8004	·7490	1·335	1·249	1·668	10′
37°00′	·6018	·7986	·7536	1·327	1·252	1·662	53°00′
10′	·6041	·7969	·7581	1·319	1·255	1·655	50′
20′	·6065	·7951	·7627	1·311	1·258	1·649	40′
30′	·6088	·7934	·7673	1·303	1·260	1·643	30′
40′	·6111	·7916	·7720	1·295	1·263	1·636	20′
50′	·6134	·7898	·7766	1·288	1·266	1·630	10′
38°00′	·6157	·7880	·7813	1·280	1·269	1·624	52°00′
10′	·6180	·7862	·7860	1·272	1·272	1·618	50′
20′	·6202	·7844	·7907	1·265	1·275	1·612	40′
30′	·6225	·7826	·7954	1·257	1·278	1·606	30′
40′	·6248	·7808	·8002	1·250	1·281	1·601	20′
50′	·6271	·7790	·8050	1·242	1·284	1·595	10′
39°00′	·6293	·7771	·8098	1·235	1·287	1·589	51°00′
10′	·6316	·7753	·8146	1·228	1·290	1·583	50′
20′	·6338	·7735	·8195	1·220	1·293	1·578	40′
30′	·6361	·7716	·8243	1·213	1·296	1·572	30′
40′	·6383	·7698	·8292	1·206	1·299	1·567	20′
50′	·6406	·7679	·8342	1·199	1·302	1·561	10′
Angle	cos	sin	cot	tan	csc	sec	Angle

VALUES OF TRIGONOMETRIC FUNCTIONS *(Concluded)*

Angle	sin	cos	tan	cot	sec	csc	Angle
40°00′	·6428	·7660	·8391	1·192	1·305	1·556	50°00′
10′	·6450	·7642	·8441	1·185	1·309	1·550	50′
20′	·6472	·7623	·8491	1·178	1·312	1·545	40′
30′	·6494	·7604	·8541	1·171	1·315	1·540	30′
40′	·6517	·7585	·8591	1·164	1·318	1·535	20′
50′	·6539	·7566	·8642	1·157	1·322	1·529	10′
41°00′	·6561	·7547	·8693	1·150	1·325	1·524	49°00′
10′	·6583	·7528	·8744	1·144	1·328	1·519	50′
20′	·6604	·7509	·8796	1·137	1·332	1·514	40′
30′	·6626	·7490	·8847	1·130	1·335	1·509	30′
40′	·6648	·7470	·8899	1·124	1·339	1·504	20′
50′	·6670	·7451	·8952	1·117	1·342	1·499	10′
42°00′	·6691	·7431	·9004	1·111	1·346	1·494	48°00′
10′	·6713	·7412	·9057	1·104	1·349	1·490	50′
20′	·6734	·7392	·9110	1·098	1·353	1·485	40′
30′	·6756	·7373	·9163	1·091	1·356	1·480	30′
40′	·6777	·7353	·9217	1·085	1·360	1·476	20′
50′	·6799	·7333	·9271	1·079	1·364	1·471	10′
43°00′	·6820	·7314	·9325	1·072	1.367	1·466	47°00′
10′	·6841	·7294	·9380	1·066	1·371	1·462	50′
20′	·6862	·7274	·9435	1·060	1·375	1·457	40′
30′	·6884	·7254	·9490	1·054	1·379	1·453	30′
40′	·6905	·7234	·9545	1·048	1·382	1·448	20′
50′	·6926	·7214	·9601	1·042	1·386	1·444	10′
44°00′	·6947	·7193	·9657	1·036	1·390	1·440	46°00′
10′	·6967	·7173	·9713	1·030	1·394	1·435	50′
20′	·6988	·7153	·9770	1·024	1·398	1·431	40′
30′	·7009	·7133	·9827	1·018	1·402	1·427	30′
40′	·7030	·7112	·9884	1·012	1·406	1·423	20′
50′	·7050	·7092	·9942	1·006	1·410	1·418	10′
45°00′	·7071	·7071	1.000	1·000	1·414	1·414	45°00′
Angle	cos	sin	cot	tan	csc	sec	Angle

TABLE 2

Common Logarithms

N	0	1	2	3	4	5	6	7	8	9
10	·0000	·0043	·0086	·0128	·0170	·0212	·0253	·0294	·0334	·0374
11	·0414	·0453	·0492	·0531	·0569	·0607	·0645	·0682	·0719	·0755
12	·0792	·0828	·0864	·0899	·0934	·0969	·1004	·1038	·1072	·1106
13	·1139	·1173	·1206	·1239	·1271	·1303	·1335	·1367	·1399	·1430
14	·1461	·1492	·1523	·1553	·1584	·1614	·1644	·1673	·1703	·1732
15	·1761	·1790	·1818	·1847	·1875	·1903	·1931	·1959	·1987	·2014
16	·2041	·2068	·2095	·2122	·2148	·2175	·2201	·2227	·2253	·2279
17	·2304	·2330	·2355	·2380	·2405	·2430	·2455	·2480	·2504	·2529
18	·2553	·2577	·2601	·2625	·2648	·2672	·2695	·2718	·2742	·2765
19	·2788	·2810	·2833	·2856	·2878	·2900	·2923	·2945	·2967	·2989
20	·3010	·3032	·3054	·3075	·3096	·3118	·3139	·3160	·3181	·3201
21	·3222	·3243	·3263	·3284	·3304	·3324	·3345	·3365	·3385	·3404
22	·3424	·3444	·3464	·3483	·3502	·3522	·3541	·3560	·3579	·3598
23	·3617	·3636	·3655	·3674	·3692	·3711	·3729	·3747	·3766	·3784
24	·3802	·3820	·3838	·3856	·3874	·3892	·3909	·3927	·3945	·3962
25	·3979	·3997	·0414	·4031	·4048	·4065	·4082	·4099	·4116	·4133
26	·4150	·4166	·4183	·4200	·4216	·4232	·4249	·4265	·4281	·4298
27	·4314	·4330	·4346	·4362	·4378	·4393	·4409	·4425	·4440	·4456
28	·4472	·4487	·4502	·4518	·4533	·4548	·4564	·4579	·4594	·4609
29	·4624	·4639	·4654	·4669	·4683	·4698	·4713	·4728	·4742	·4757
30	·4771	·4786	·4800	·4814	·4829	·4843	·4857	·4871	·4886	·4900
31	·4914	·4928	·4942	·4955	·4969	·4983	·4997	·5011	·5024	·5038
32	·5051	·5065	·5079	·5092	·5105	·5119	·5132	·5145	·5159	·5172
33	·5185	·5198	·5211	·5224	·5237	·5250	·5263	·5276	·5289	·5302
34	·5315	·5328	·5340	·5353	·5366	·5378	·5391	·5403	·5416	·5428
35	·5441	·5453	·5465	·5478	·5490	·5502	·5514	·5527	·5539	·5551
36	·5563	·5575	·5587	·5599	·5611	·5623	·5635	·5647	·5658	·5670
37	·5682	·5694	·5705	·5717	·5729	·5740	·5752	·5763	·5775	·5786
38	·5798	·5809	·5821	·5832	·5843	·5855	·5866	·5877	·5888	·5899
39	·5911	·5922	·5933	·5944	·5955	·5966	·5977	·5988	·5999	·6010
40	·6021	·6031	·6042	·6053	·6064	·6075	·6085	·6096	·6107	·6117
41	·6128	·6138	·6149	·6160	·6170	·6180	·6191	·6201	·6212	·6222
42	·6232	·6243	·6253	·6263	·6274	·6284	·6294	·6304	·6314	·6325
43	·6335	·6345	·6355	·6365	·6375	·6385	·6395	·6405	·6415	·6425
44	·6435	·6444	·6454	·6464	·6474	·6484	·6493	·6503	·6513	·6522
45	·6532	·6542	·6551	·6561	·6571	·6580	·6590	·6599	·6609	·6618
46	·6628	·6637	·6646	·6656	·6665	·6675	·6684	·6693	·6702	·6712
47	·6721	·6730	·6739	·6749	·6758	·6767	·6776	·6785	·6794	·6803
48	·6812	·6821	·6830	·6839	·6848	·6857	·6866	·6875	·6884	·6893
49	·6802	·6911	·6920	·6928	·6937	·6946	·6955	·6964	·6972	·6891
50	·6990	·6998	·7007	·7016	·7024	·7033	·7042	·7050	·7059	·7067
51	·7076	·7084	·7093	·7101	·7110	·7118	·7126	·7135	·7143	·7152
52	·7160	·7168	·7177	·7185	·7193	·7202	·7210	·7218	·7226	·7235
53	·7243	·7251	·7259	·7267	·7275	·7284	·7292	·7300	·7308	·7316
54	·7324	·7332	·7340	·7348	·7356	·7364	·7372	·7380	·7388	·7396

COMMON LOGARITHMS *(Continued)*

N	0	1	2	3	4	5	6	7	8	9
55	·7404	·7412	·7419	·7427	·7435	·7443	·7451	·7459	·7466	·7474
56	·7482	·7490	·7497	·7505	·7513	·7520	·7528	·7536	·7543	·7551
57	·7559	·7566	·7574	·7582	·7589	·7597	·7604	·7612	·7619	·7627
58	·7634	·7642	·7649	·7657	·7664	·7672	·7679	·7686	·7694	·7701
59	·7709	·7716	·7723	·7731	·7738	·7745	·7752	·7760	·7767	.7774
60	·7782	·7789	·7796	·7803	·7810	·7818	·7825	·7832	·7839	·7846
61	·7853	·7860	·7868	·7875	·7882	·7889	·7896	·7903	·7910	·7917
62	·7924	·7931	·7938	·7945	·7952	·7959	·7966	·7973	·7980	·7987
63	·7993	·8000	·8007	·8014	·8021	·8028	·8035	·8041	·8048	·8055
64	·8062	·8069	·8075	·8082	·8089	·8096	·8102	·8109	·8106	·8122
65	·8129	·8136	·8142	·8149	·8156	·8162	·8169	·8176	·8182	·8189
66	·8195	·8202	·8209	·8215	·8222	·8228	·8235	·8241	·8248	·8254
67	·8261	·8267	·8274	·8280	·8287	·8293	·8299	·8306	·8312	·8319
68	·8325	·8331	·8338	·8344	·8351	·8357	·8363	·8370	·8376	·8332
69	·8388	·8395	·8401	·8407	·8414	·8420	·8426	·8432	·8439	·8444
70	·8451	·8457	·8463	·8470	·8476	·8482	·8488	·8494	·8500	·8505
71	·8513	·8519	·8525	·8531	·8537	·8543	·8549	·8555	·8561	·8567
72	·8573	·8579	·8585	·8591	·8597	·8603	·8609	·8615	·8621	·8627
73	·8633	·8639	·8645	·8651	·8657	·8663	·8669	·8675	·8681	·8686
74	·8692	·8698	·8704	·8710	·8716	·8722	·8727	·8733	·8739	.8745
75	·8751	·8756	·8762	·8768	·8774	·8779	·8785	·8791	·8797	·8802
76	·8808	·8814	·8820	·8825	·8831	·8837	·8842	·8848	·8854	·8859
77	·8865	·8871	·8876	·8882	·8887	·8893	·8899	·8904	·8910	·8915
78	·8921	·8927	·8932	·8938	·8943	·8949	·8954	·8960	·8956	·8971
79	·8976	·8982	·8987	·8993	·8998	·9004	·9009	·9015	·9020	·9025
80	·9031	·9036	·9042	·9047	·9053	·9058	·9063	·9069	·9074	.9079
81	·9085	·9090	·9096	·9101	·9106	·9112	·9117	·9122	·9128	.9133
82	·9138	·9143	·9149	·9154	·9159	·9165	·9170	·9175	·9180	·9186
83	·9191	·9196	·9201	·9206	·9212	·9217	·9222	·9227	·9232	·9238
84	·9243	·9248	·9253	.9258	·9263	·9269	·9274	·9279	·9284	·9289
85	·9294	·9299	·9304	·9309	·9315	·9320	·9325	·9330	·9335	·9340
86	·9345	·9350	·9355	·9360	·9365	·9370	·9375	·9380	·9385	·9390
87	·9395	·9400	·9405	·9410	·9415	·9420	·9425	·9430	·9435	·9440
88	·9445	·9450	·9455	·9460	·9465	·9469	·9474	·9479	·9484	·9489
89	·9494	·9499	·9504	·9509	·9513	·9518	·9523	·9528	·9533	·9538
90	·9542	·9547	·9552	·9557	·9562	·9566	·9571	·9576	·9581	·9586
91	·9590	·9595	·9600	·9605	·9609	·9614	·9619	·9624	·9628	·9633
92	·9638	·9643	·9647	·9652	·9657	·9661	·9666	·9671	·9675	·9680
93	·9685	·9689	·9694	·9699	·9703	·9708	·9713	·9717	·9722	·9727
94	·9731	·9736	·9741	·9745	·9750	·9754	·9759	·9763	·9768	·9773
95	·9777	·9782	·9786	·9791	·9795	·9800	·9805	·9809	·9814	·9818
96	·9823	·9827	·9832	·9836	·9841	·9845	·9850	·9854	·9859	·9863
97	·9868	·9872	·9877	·9881	·9886	·9890	·9894	·9899	·9903	·9908
98	·9912	·9917	·9921	·9926	·9930	·9934	·9939	·9943	·9948	·9952
99	·9956	·9961	·9965	·9969	·9974	·9978	·9983	·9987	·9991	·9996

TABLE 3

FOUR-PLACE LOGARITHMS OF VALUES OF TRIGONOMETRIC FUNCTIONS*

Angle	L sin	L tan	L cot	L cos	Angle
0°00′	—	—	—	10·0000	90°00′
10′	7·4637	7·4637	12·5363	10·0000	50′
20′	7·7648	7·7648	12·2352	10·0000	40′
30′	7·9408	7·9409	12·0591	10·0000	30′
40′	8·0658	8·0658	11·9342	10·0000	20′
50′	8·1627	8·1627	11·8373	10·0000	10′
1°00′	8·2419	8·2419	11·7581	9·9999	89°00′
10′	8·3088	8·3089	11·6911	9·9999	50′
20′	8·3668	8·3669	11·6331	9·9999	40′
30′	8·4179	8·4181	11·5819	9·9999	30′
40′	8·4637	8·4638	11·5362	9·9998	20′
50′	8·5050	8·5053	11·4947	9·9998	10′
2°00′	8·5428	8·5431	11·4569	9·9997	88°00′
10′	8·5776	8·5779	11·4221	9·9997	50′
20′	8·6097	8·6101	11·3899	9·9996	40′
30′	8·6397	8·6401	11·3599	9·9996	30′
40′	8·6677	8·6682	11·3318	9·9995	20′
50′	8·6940	8·6945	11·3055	9·9995	10′
3°00′	8·7188	8·7194	11·2806	9·9994	87°00′
10′	8·7423	8·7429	11·2571	9·9993	50′
20′	8·7645	8·7652	11·2348	9·9993	40′
30′	8·7857	8·7865	11·2135	9·9992	30′
40′	8·8059	8·8067	11·1933	9·9991	20′
50′	8·8251	8·8261	11·1739	9·9990	10′
4°00′	8·8436	8·8446	11·1554	9·9989	86°00′
10′	8·8613	8·8624	11·1376	9·9989	50′
20′	8·8783	8·8795	11·1205	9·9988	40′
30′	8·8946	8·8960	11·1040	9·9987	30′
40′	8·9104	8·9118	11·0882	9·9986	20′
50′	8·9256	8·9272	11·0728	9·9985	10′
5°00′	8·9403	8·9420	11·0580	9·9983	85°00′
10′	8·9545	8·9563	11·0437	9·9982	50′
20′	8·9682	8·9701	11·0299	9·9981	40′
30′	8·9816	8·9836	11·0164	9·9980	30′
40′	8·9945	8·9966	11·0034	9·9979	20′
50′	9·0070	9·0093	10·9907	9·9977	10′
6°00′	9·0192	9·0216	10·9784	9·9976	84°00′
10′	9·0311	9·0336	10·9664	9·9975	50′
20′	9·0426	9·0453	10·9547	9·9973	40′
30′	9·0539	9·0567	10·9433	9·9972	30′
40′	9·0648	9·0678	10·9322	9·9971	20′
50′	9·0755	9·0786	10·9214	9·9969	10′
7°00′	9·0859	9·0891	10·9109	9·9968	83°00′
10′	9·0961	9·0995	10·9005	9·9966	50′
20′	9·1060	9·1096	10·8904	9·9964	40′
30′	9·1157	9·1194	10·8806	9·9963	30′
40′	9·1252	9·1291	10·8709	9·9961	20′
50′	9·1345	9·1385	10·8615	9·9959	10′
8°00′	9·1436	9·1478	10·8522	9·9958	82°00′
10′	9·1525	9·1569	10·8431	9·9956	50′
20′	9·1612	9·1658	10·8342	9·9954	40′
30′	9·1697	9·1745	10·8255	9·9952	30′
40′	9·1781	9·1831	10·8169	9·9950	20′
50′	9·1863	9·1915	10·8085	9·9948	10′
9°00′	9·1943	9·1997	10·8003	9·9946	81°00′
Angle	L cos	L cot	L tan	L sin	Angle

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* This table gives the logarithms increased by 10. Hence in each case 10 should be subtracted.

FOUR-PLACE LOGARITHMS OF VALUES OF TRIGONOMETRIC FUNCTIONS

Angle	L sin	L tan	L cot	L cos	Angle
9°00′	9·1943	9·1997	10·8003	9·9946	81°00′
10′	9·2022	9·2078	10·7922	9·9944	50′
20′	9·2100	9·2158	10·7842	9·9942	40′
30′	9·2176	9·2236	10·7764	9·9940	30′
40′	9·2251	9·2313	10·7687	9·9938	20′
50′	9·2324	9·2389	10·7611	9·9936	10′
10°00′	9·2397	9·2463	10·7537	9·9934	80°00′
10′	9·2468	9·2536	10·7464	9·9931	50′
20′	9·2538	9·2609	10·7391	9·9929	40′
30′	9·2606	9·2680	10·7320	9·9927	30′
40′	9·2674	9·2750	10·7250	9·9924	20′
50′	9·2740	9·2819	10·7181	9·9922	10′
11°00′	9·2806	9·2887	10·7113	9·9919	79°00′
10′	9·2870	9·2953	10·7047	9·9917	50′
20′	9·2934	9·3020	10·6980	9·9914	40′
30′	9·2997	9·3085	10·6915	9·9912	30′
40′	9·3058	9·3149	10·6851	9·9909	20′
50′	9·3119	9·3112	10·6788	9·9907	10′
12°00′	9·3179	9·3275	10·6725	9·9904	78°00′
10′	9·3238	9·3336	10·6664	9·9901	50′
20′	9·3296	9·3397	10·6603	9·9899	40′
30′	9·3353	9·3458	10·6542	9·9896	30′
40′	9·3410	9·3517	10·6483	9·9893	20′
50′	9·3466	9·3576	10·6424	9·9890	10′
13°00′	9·3521	9·3634	10·6366	9·9887	77°00′
10′	9·3575	9·3691	10·6309	9·9884	50′
20′	9·3629	9·3748	10·6252	9·9881	40′
30′	9·3682	9·3804	10·6196	9·9878	30′
40′	9·3734	9·3859	10·6141	9·9875	20′
50′	9·3786	9·3914	10·6086	9·9872	10′
14°00′	9·3837	9·3968	10·6032	9·9869	76°00′
10′	9·3887	9·4021	10·5979	9·9866	50′
20′	9·3937	9·4074	10·5926	9·9863	40′
30′	9·3986	9·4127	10·5873	9·9859	30′
40′	9·4035	9·4178	10·5822	9·9856	20′
50′	9·4083	9·4230	10·5770	9·9853	10′
15°00′	9·4130	9·4281	10·5719	9·9849	75°00′
10′	9·4177	9·4331	10·5669	9·9846	50′
20′	9·4223	9·4381	10·5619	9·9843	40′
30′	9·4269	9·4430	10·5570	9·9839	30′
40′	9·4314	9·4479	10·5521	9·9836	20′
50′	9·4359	9·4527	10·5473	9·9832	10′
16°00′	9·4403	9·4575	10·5425	9·9828	74°00′
10′	9·4447	9·4622	10·5378	9·9825	50′
20′	9·4491	9·4669	10·5331	9·9821	40′
30′	9·4533	9·4716	10·5284	9·9817	30′
40′	9·4576	9·4762	10·5238	9·9814	20′
50′	9·4618	9·4808	10·5192	9·9810	10′
17°00′	9·4659	9·4853	10·5147	9·9806	73°00′
10′	9·4700	9·4898	10·5102	9·9802	50′
20′	9·4741	9·4943	10·5057	9·9798	40′
30′	9·4781	9·4987	10·5013	9·9794	30′
40′	9·4821	9·5031	10·4969	9·9790	20′
50′	9·4861	9·5075	10·4925	9·9786	10′
18°00′	9·4900	9·5118	10·4882	9·9782	72°00′
Angle	L cos	L cot	L tan	L sin	Angle

FOUR-PLACE LOGARITHMS OF VALUES OF TRIGONOMETRIC FUNCTIONS

Angle	L sin	L tan	L cot	L cos	Angle
18°00′	9·4900	9·5118	10·4882	9·9782	72°00′
10′	9·4939	9·5161	10·4839	9·9778	50′
20′	9·4977	9·5203	10·4797	9·9774	40′
30′	9·5015	9·5245	10·4755	9·9770	30′
40′	9·5052	9·5287	10·4713	9·9765	20′
50′	9·5090	9·5329	10·4671	9·9761	10′
19°00′	9·5126	9·5370	10·4630	9·9757	71°00′
10′	9·5163	9·5411	10·4589	9·9752	50′
20′	9·5199	9·5451	10·4549	9·9748	40′
30′	9·5235	9·5491	10·4509	9·9743	30′
40′	9·5270	9·5531	10·4469	9·9739	20′
50′	9·5306	9·5571	10·4429	9·9734	10′
20°00′	9·5341	9·5611	10·4389	9·9730	70°00′
10′	9·5375	9·5650	10·4350	9·9725	50′
20′	9·5409	9·5689	10·4311	9·9721	40′
30′	9·5443	9·5727	10·4273	9·9716	30′
40′	9·5477	9·5766	10·4234	9·9711	20′
50′	9·5510	9·5804	10·4196	9·9706	10′
21°00′	9·5543	9·5842	10·4158	9·9702	69°00′
10′	9·5576	9·5879	10·4121	9·9697	50′
20′	9·5609	9·5917	10·4083	9·9692	40′
30′	9·5641	9·5954	10·4046	9·9687	30′
40′	9·5673	9·5991	10·4009	9·9682	20′
50′	9·5704	9·6028	10·3972	9·9677	10′
22°00′	9·5736	9·6064	10·3936	9·9672	68°00′
10′	9·5767	9·6100	10·3900	9·9667	50′
20′	9·5798	9·6136	10·3864	9·9661	40′
30′	9·5828	9·6172	10·3828	9·9656	30′
40′	9·5859	9·6208	10·3792	9·9651	20′
50′	9·5889	9·6243	10·3757	9·9646	10′
23°00′	9·5919	9·6279	10·3721	9·9640	67°00′
10′	9·5948	9·6314	10·3686	9·9635	50′
20′	9·5978	9·6348	10·3652	9·9629	40′
30′	9·6007	9·6383	10·3617	9·9624	30′
40′	9·6036	9·6417	10·3583	9·9618	20′
50′	9·6065	9·6452	10·3548	9·9613	10′
24°00′	9·6093	9·6486	10·3514	9·9607	66°00′
10′	9·6121	9·6520	10·3480	9·9602	50′
20′	9·6149	9·6553	10·3447	9·9596	40′
30′	9·6177	9·6587	10·3413	9·9590	30′
40′	9·6205	9·6620	10·3380	9·9584	20′
50′	9·6232	9·6654	10·3346	9·9579	10′
25°00′	9·6259	9·6687	10·3313	9·9573	65°00′
10′	9·6286	9·6720	10·3280	9·9567	50′
20′	9·6313	9·6752	10·3248	9·9561	40′
30′	9·6340	9·6785	10·3215	9·9555	30′
40′	9·6366	9·6817	10·3183	9·9549	20′
50′	9·6392	9·6850	10·3150	9·9543	10′
26°00′	9·6418	9·6882	10·3118	9·9537	64°00′
10′	9·6444	9·6914	10·3086	9·9530	50′
20′	9·6470	9·6946	10·3054	9·9524	40′
30′	9·6495	9·6977	10·3023	9·9518	30′
40′	9·6521	9·7009	10·2991	9·9512	20′
50′	9·6546	9·7040	10·2960	9·9505	10′
27°00′	9·6570	9·7072	10·2928	9·9499	63°00′
Angle	L cos	L cot	L tan	L sin	Angle

Four-place Logarithms of Values of Trigonometric Functions

Angle	L sin	L tan	L cot	L cos	Angle
27°00′	9·6570	9·7072	10·2928	9·9499	63°00′
10′	9·6595	9·7103	10·2897	9·9492	50′
20′	9·6620	9·7134	10·2866	9·9486	40′
30′	9·6644	9·7165	10·2835	9·9479	30′
40′	9·6668	9·7196	10·2804	9·9473	20′
50′	9·6692	9·7226	10·2774	9·9466	10′
28°00′	9·6716	9·7257	10·2743	9·9459	62°00′
10′	9·6740	9·7287	10·2713	9·9453	50′
20′	9·6763	9·7317	10·2683	9·9446	40′
30′	9·6787	9·7348	10·2652	9·9439	30′
40′	9·6810	9·7378	10·2622	9·9432	20′
50′	9·6833	9·7408	10·2592	9·9425	10′
29°00′	9·6856	9·7438	10·2562	9·9418	61°00′
10′	9·6878	9·7467	10·2533	9·9411	50′
20′	9·6901	9·7497	10·2503	9·9404	40′
30′	9·6923	9·7526	10·2474	9·9397	30′
40′	9·6946	9·7556	10·2444	9·9390	20′
50′	9·6968	9·7585	10·2415	9·9383	10′
30°00′	9·6990	9·7614	10·2386	9·9375	60°00′
10′	9·7012	9·7644	10·2356	9·9368	50′
20′	9·7033	9·7673	10·2327	9·9361	40′
30′	9·7055	9·7701	10·2299	9·9353	30′
40′	9·7076	9·7730	10·2270	9·9346	20′
50′	9·7097	9·7759	10·2241	9·9338	10′
31°00′	9·7118	9·7788	10·2212	9·9331	59°00′
10′	9·7139	9·7816	10·2184	9·9323	50′
20′	9·7160	9·7845	10·2155	9·9315	40′
30′	9·7181	9·7873	10·2127	9·9308	30′
40′	9·7201	9·7902	10·2098	9·9300	20′
50′	9·7222	9·7930	10·2070	9·9292	10′
32°00′	9·7242	9·7958	10·2042	9·9284	58°00′
10′	9·7262	9·7986	10·2014	9·9276	50′
20′	9·7282	9·8014	10·1986	9·9268	40′
30′	9·7302	9·8042	10·1958	9·9260	30′
40′	9·7322	9·8070	10·1930	9·9252	20′
50′	9·7342	9·8097	10·1903	9·9244	10′
33°00′	9·7361	9·8125	10·1875	9·9236	57°00′
10′	9·7380	9·8153	10·1847	9·9228	50′
20′	9·7400	9·8180	10·1820	9·9219	40′
30′	9·7419	9·8208	10·1792	9·9211	30′
40′	9·7438	9·8235	10·1765	9·9203	20′
50′	9·7457	9·8263	10·1737	9·9194	10′
34°00′	9·7476	9·8290	10·1710	9·9186	56°00′
10′	9·7494	9·8317	10·1683	9·9177	50′
20′	9·7513	9·8344	10·1656	9·9169	40′
30′	9·7531	9·8371	10·1629	9·9160	30′
40′	9·7550	9·8398	10·1602	9·9151	20′
50′	9·7568	9·8425	10·1575	9·9142	10′
35°00′	9·7586	9·8452	10·1548	9·9134	55°00′
10′	9·7604	9·8479	10·1521	9·9125	50′
20′	9·7622	9·8506	10·1494	9·9116	40′
30′	9·7640	9·8533	10·1467	9·9107	30′
40′	9·7657	9·8559	10·1441	9·9098	20′
50′	9·7675	9·8586	10·1414	9·9089	10′
Angle	L cos	L cot	L tan	L sin	Angle

Four-place Logarithms of Values of Trigonometric Functions

Angle	L sin	L tan	L cot	L cos	Angle
36°00′	9·7692	9·8613	10·1387	9·9080	54°00′
10′	9·7710	9·8639	10·1361	9·9070	50′
20′	9·7727	9·8666	10·1334	9·9061	40′
30′	9·7744	9·8692	10·1308	9·9052	30′
40′	9·7761	9·8718	10·1282	9·9042	20′
50′	9·7778	9·8745	10·1255	9·9033	10′
37°00′	9·7795	9·8771	10·1229	9·9023	53°00′
10′	9·7811	9·8797	10·1203	9·9014	50′
20′	9·7828	9·8824	10·1176	9·9004	40′
30′	9·7844	9·8850	10·1150	9·8995	30′
40′	9·7861	9·8876	10·1124	9·8985	20′
50′	9·7877	9·8902	10·1098	9·8975	10′
38°00′	9·7893	9·8928	10·1072	9·8965	52°00′
10′	9·7910	9·8945	10·1046	9·8955	50′
20′	9·7926	9·8980	10·1020	9·8945	40′
30′	9·7941	9·9006	10·0994	9·8935	30′
40′	9·7957	9·9032	10·0968	9·8925	20′
50′	9·7973	9·9058	10·0942	9·8915	10′
39°00′	9·7989	9·9084	10·0916	9·8905	51°00′
10′	9·8004	9·9110	10·0890	9·8895	50′
20′	9·8020	9·9135	10·0865	9·8884	40′
30′	9·8035	9·9161	10·0839	9·8874	30′
40′	9·8050	9·9187	10·0813	9·8864	20′
50′	9·8066	9·9212	10·0788	9·8853	10′
40°00′	9·8081	9·9238	10·0762	9·8843	50°00′
10′	9·8096	9·9264	10·0736	9·8832	50′
20′	9·8111	9·9289	10·0711	9·8821	40′
30′	9·8125	9·9315	10·0685	9·8810	30′
40′	9·8140	9·9341	10·0659	9·8800	20′
50′	9·8155	9·9366	10·0634	9·8789	10′
41°00′	9·8169	9·9392	10·0608	9·8778	49°00′
10′	9·8184	9·9417	10·0583	9·8767	50′
20′	9·8198	9·9443	10·0557	9·8756	40′
30′	9·8213	9·9468	10·0532	9·8745	30′
40′	9·8227	9·9494	10·0506	9·8733	20′
50′	9·8241	9·9519	10·0481	9·8722	10′
42°00′	9·8255	9·9544	10·0456	9·8711	48°00′
10′	9·8269	9·9570	10·0430	9·8699	50′
20′	9·8283	9·9595	10·0405	9·8688	40′
30′	9·8297	9·9621	10·0379	9·8676	30′
40′	9·8311	9·9646	10·0354	9·8665	20′
50′	9·8324	9·9671	10·0329	9·8653	10′
43°00′	9·8338	9·9697	10·0303	9·8641	47°00′
10′	9·8351	9·9722	10·0278	9·8629	50′
20′	9·8365	9·9747	10·0253	9·8618	40′
30′	9·8378	9·9772	10·0228	9·8606	30′
40′	9·8391	9·9798	10·0202	9·8594	20′
50′	9·8405	9·9823	10·0177	9·8582	10′
44°00′	9·8418	9·9848	10·0152	9·8569	46°00′
10′	9·8431	9·9874	10·0126	9·8557	50′
20′	9·8444	9·9899	10·0101	9·8545	40′
30′	9·8457	9·9924	10·0076	9·8532	30′
40′	9·8469	9·9949	10·0051	9·8520	20′
50′	9·8482	9·9975	10·0025	9·8507	10′
45°00′	9·8495	10·0000	10·0000	9·8495	45°00′
Angle	L cos	L cot	L tan	L sin	Angle

TABLE 4

SQUARES AND SQUARE ROOTS

N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N	N^2	$\sqrt{N}$	$\sqrt{10N}$
1·0	1·00	1.000	3·162	5·5	30·25	2·345	7·416
1·1	1·21	1·049	3·317	5·6	31·36	2·366	7·483
1·2	1·44	1·095	3·464	5·7	32·49	2·387	7·550
1·3	1·69	1·140	3·606	5·8	33·64	2·408	7·616
1·4	1·96	1·183	3·742	5.9	34·81	2·429	7·681
1·5	2·25	1·225	3·873	6·0	36·00	2·449	7·746
1·6	2·56	1·265	4·000	6·1	37·21	2·470	7·810
1·7	2·89	1·304	4·123	6·2	38·44	2·490	7·870
1·8	3·24	1·342	4·243	6·3	39·69	2·510	7·937
1·9	3·61	1·378	4·359	6·4	40·96	2·530	8·000
2·0	4·00	1·414	4·472	6·5	42·25	2·550	8·062
2·1	4·41	1·449	4·583	6·6	43·56	2·569	8·124
2·2	4·84	1·483	4·690	6·7	44·89	2·588	8·185
2·3	5·29	1·517	4·796	6·8	46·24	2·608	8·246
2·4	5·76	1·549	4·899	6·9	47·61	2·627	8·307
2·5	6·25	1·581	5·000	7·0	49·00	2·646	8·367
2·6	6·76	1·612	5·099	7·1	50·41	2·665	8·426
2·7	7·29	1·643	5·196	7·2	51·84	2·683	8·485
2·8	7·84	1·673	5·292	7·3	53·29	2·702	8·544
2·9	8·41	1·703	5·385	7·4	54·76	2·720	8·602
3·0	9·00	1·732	5·477	7·5	56·25	2·739	8·660
3·1	9·61	1·761	5·568	7·6	57·76	2·757	8·718
3·2	10·24	1·789	5·657	7·7	59·29	2·775	8·775
3·3	10·89	1·817	5·745	7·8	60·84	2·793	8·832
3·4	11·56	1·844	5·831	7·9	62·41	2·811	8·888
3·5	12·25	1·871	5·916	8·0	64·00	2·828	8·944
3·6	12·96	1·897	6·000	8·1	65·61	2·846	9·000
3·7	13·69	1·924	6·083	8·2	67·24	2·864	9·055
3·8	14·44	1·949	6·164	8·3	68·89	2·881	9·110
3·9	15·21	1·975	6·245	8·4	70·56	2·898	9·165
4·0	16·00	2·000	6·325	8·5	72·25	2·915	9·220
4·1	16·81	2·025	6·403	8·6	73·96	2·933	9·274
4·2	17·64	2·049	6·481	8·7	75·69	2·950	9·327
4·3	18·49	2·074	6·557	8·8	77·44	2·966	9·381
4·4	19·36	2·098	6·633	8·9	79·21	2·983	9·434
4·5	20·25	2·121	6·708	9·0	81·00	3·000	9·487
4·6	21·16	2·145	6·782	9·1	82·81	3·017	9·539
4·7	22·09	2·168	6·856	9·2	84·64	3·033	9·592
4·8	23·04	2·191	6·928	9·3	86·49	3·050	9·644
4·9	24·01	2·214	7·000	9·4	88·36	3·066	9·695
5·0	25·00	2·236	7·071	9·5	90·25	3·082	9·747
5·1	26·01	2·258	7·141	9·6	92·16	3·098	9·798
5·2	27·04	2·280	7·211	9·7	94·09	3·114	9·849
5·3	28·09	2·302	7·280	9·8	96·04	3·130	9·899
5·4	29·16	2·324	7·348	9·9	98·01	3·146	9·950
				10·0	100·00	3·162	1·000

TABLE 5

Cubes and Cube Roots

N	N^3	$\sqrt[3]{N}$	$\sqrt[3]{10N}$	$\sqrt[3]{100N}$	N	N^3	$\sqrt[3]{N}$	$\sqrt[3]{10N}$	$\sqrt[3]{100N}$
1·0	1·000	1·000	2·154	4·642	5·5	166·375	1·765	3·803	8·193
1·1	1·331	1·032	2·224	4·791	5·6	175·616	1·776	3·826	8·243
1·2	1·728	1·063	2·289	4·932	5·7	185·193	1·786	3·849	8·291
1·3	2·197	1·091	2·351	5·066	5·8	195·112	1·797	3·871	8·340
1·4	2·744	1·119	2·410	5·192	5·9	205·379	1·807	3·893	8·387
1·5	3·375	1·145	2·466	5·313	6·0	216·000	1·817	3·915	8·434
1·6	4·096	1·170	2·520	5·429	6·1	226·981	1·827	3·936	8·481
1·7	4·913	1·193	2·571	5·540	6·2	238·328	1·837	3·958	8·527
1·8	5·832	1·216	2·621	5·646	6·3	250·047	1·847	3·979	8·573
1·9	6·859	1·239	2·668	5·749	6·4	262·144	1·857	4·000	8·618
2·0	8·000	1·260	2·714	5·848	6·5	274·625	1·866	4·021	8·662
2·1	9·261	1·281	2·759	5·944	6·6	287·496	1·876	4·041	8·707
2·2	10·648	1·301	2·802	6·037	6·7	300·763	1·885	4·062	8·750
2·3	12·167	1·320	2·844	6·127	6·8	314·432	1·895	4·082	8·794
2·4	13·824	1·339	2·884	6·214	6·9	328·509	1·904	4·102	8·837
2·5	15·625	1·357	2·924	6·300	7·0	343·000	1·913	4·121	8·879
2·6	17·576	1·375	2·962	6·383	7·1	357·911	1·922	4·141	8·921
2·7	19·683	1·392	3·000	6·463	7·2	373·248	1·931	4·160	8·963
2·8	21·952	1·409	3·037	6·542	7·3	389·017	1·940	4·179	9·004
2·9	24·389	1·426	3·072	6·619	7·4	405·224	1·949	4·198	9·045
3·0	27·000	1·442	3·107	6·694	7·5	421·875	1·957	4·217	9·086
3·1	29·791	1·458	3·141	6·768	7·6	438·976	1·966	4·236	9·126
3·2	32·768	1·474	3·175	6·840	7·7	456·533	1·975	4·254	9·166
3·3	35·937	1·489	3·208	6·910	7·8	474·552	1·983	4·273	9·205
3·4	39·304	1·504	3·240	6·980	7·9	493·039	1·992	4·291	9·244
3·5	42·875	1·518	3·271	7·047	8·0	512·000	2·000	4·309	9·283
3·6	46·656	1·533	3·302	7·114	8·1	531·441	2·008	4·327	9·322
3·7	50·653	1·547	3·332	7·179	8·2	551·368	2·017	4·344	9·360
3·8	54·872	1·560	3·362	7·243	8·3	571·787	2·025	4·362	9·398
3·9	59·319	1·574	3·391	7·306	8·4	592·704	2·033	4·380	9·435
4·0	64·000	1·587	3·420	7·368	8·5	614·125	2·041	4·397	9·473
4·1	68·921	1·601	3·448	7·429	8·6	636·056	2·049	4·414	9·510
4·2	74·088	1·613	3·476	7·489	8·7	658·503	2·057	4·431	9·546
4·3	79·507	1·626	3·503	7·548	8·8	681·472	2·065	4·448	9·583
4·4	85·184	1·639	3·530	7·606	8·9	704·969	2·072	4·465	9·619
4·5	91·125	1·651	3·557	7·663	9·0	729·000	2·080	4·481	9·655
4·6	97·336	1·663	3·583	7·719	9·1	753·571	2·088	4·498	9·691
4·7	103·823	1·675	3·609	7·775	9·2	778·688	2·095	4·514	9·726
4·8	110·592	1·687	3·634	7·830	9·3	804·357	2·103	4·531	9·761
4·9	117·649	1·698	3·659	7·884	9·4	830·584	2·110	4·547	9·796
5·0	125·000	1·710	3·684	7·937	9·5	857·375	2·118	4·563	9·830
5·1	132·651	1·721	3·708	7·990	9·6	884·736	2·125	4·579	9·865
5·2	140·608	1·732	3·733	8·041	9·7	912·673	2·133	4·595	9·899
5·3	148·877	1·744	3·756	8·093	9·8	941·192	2·140	4·610	9·933
5·4	157·464	1·754	3·780	8·143	9·9	970·299	2·147	4·626	9·967
					10·0	1000·000	2·154	4·642	0·000

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